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Ghost D-branes

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Abstract: We define a ghost D-brane in superstring theories as an object that cancels the effects of an ordinary D-brane. The supergroups $U(N|M)$ and $OSp(N|M)$ arise as gauge symmetries in the supersymmetric world-volume theory of D-branes and ghost D-branes. A system with a pair of D-brane and ghost D-brane located at the same location is physically equivalent to the closed string vacuum. When they are separated, the system becomes a new brane configuration. We generalize the type I/heterotic duality by including n ghost D9-branes on the type I side and by considering the heterotic string whose gauge group is $OSp(32+2n|2n)$. Motivated by the type IIB S-duality applied to D9- and ghost D9-branes, we also find type II-like closed superstrings with $U(n|n)$ gauge symmetry.

Keywords: Conformal Field Models in String Theory, Brane Dynamics in Gauge Theories, String Duality, D-branes.

Contents

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1. Introduction and summary

D-branes [1] have been the basis of most developments in string theory for the last ten years. The AdS/CFT correspondence [2], for example, was discovered by considering the near-horizon limit of D3-branes in type IIB string theory. Many of exact calculations in topological string theory have also been made possible by the inclusion of D-branes [3]. The identification of fermions with D0-branes revivided the investigation of two dimensional string theories [4–6].

In this paper, we introduce the notion of ghost D-branes in superstring theories. We define a ghost D-brane as an object that cancels the effects of an ordinary D-brane. We call them ghost D-branes because the gauge fields and transverse scalars on them have the wrong signs in their kinetic terms. The open strings between a D-brane and a ghost D-brane have the opposite statistics relative to the usual ones. Ghost D-branes replace ordinary Chan-Paton matrices with supermatrices. The Lie supergroups $U(N|M)$ and $OSp(N|M)$ arise as gauge groups in this way. In particular, this leads to a new type I string theory with gauge group $OSp(32+2n|2n)$ and type IIB string with gauge group $U(n|n)$ with any positive integer n. A similar idea can clearly be applied to define ghost M2 and M5-branes in M-theory.

We stress that ghost D-branes are different from anti D-branes. The boundary state of a ghost D-brane is minus the whole boundary state of an ordinary D-brane, whereas an anti D-brane has a minus sign just in the RR sector. Thus ghost D-branes have negative tension and anti-gravitate. A ghost D-brane completely cancels the effects of a D-brane when they are on top of each other with a trivial gauge field background. When M ghost D-branes are coincident with $N(\geq M)$ D-branes, the system is equivalent to the one with $N-M$ D-branes with no ghost branes as we show explicitly. More subtle are the situations where we separate ghost D-branes from D-branes or turn on non-trivial gauge backgrounds like Wilson lines. The supergravity solution for isolated ghost D-branes appears to be singular near the branes essentially because they are anti-gravitating sources. A system with ghost branes have two kinds of ghost-like fields: some of them have the wrong spin-statistics relations while the others have the wrong signs in their kinetic terms. The theory will not be unitary when ghost branes are not canceled by ordinary branes. Once the fields on ghost branes are excited, there may be instabilities because of the wrong signs in the kinetic terms. We expect that despite these pathologies the concept of ghost branes and their cancellation against ordinary branes will be useful tools.

Ghost D-branes preserve the same supercharges as ordinary D-branes, and it is natural to consider its strong coupling dual. Indeed we will show that the type I $OSp(32+2n|2n)$ string theory is dual to a heterotic $OSp(32+2n|2n)$ string theory (see [7] for an earlier discussion on this heterotic string) by generalizing the type I/heterotic duality [8]. We will also discuss a supergroup extension of the $E_8 \times E_8$ heterotic string. This heterotic string has infinitely many massless gauge fields. We expect that it is uplifted to the Horava-Witten [9] like setup in M-theory. It is also interesting to consider the type IIB S-duality applied to n pairs of D9- and ghost D9-branes. As its strong coupling limit, we find a novel superstrings that have the $U(n|n)$ gauge fields in the closed string sector. This superstring theory looks like a heterotic modification of the ordinary type IIB string, which is equivalent to the type IIB string with n NS9-branes and n ghost NS9-branes.

Another context where various Lie groups appear is the study of string junctions suspended between (p,q) 7-branes (refer to e.g.[10] and references therein). It is natural to add ghost 7-branes and ask which Lie supergroups can appear as symmetry groups. We will realize $SU(N|M)$ and $OSp(2M|2N)$ symmetries in this way and obtain their Dynkin diagrams in terms of string junctions.

There is a rather similar example in two dimensional string theory.[1] The $c=1$ matrix model dual provides its non-perturbative definition. This model is equivalent to the quantum mechanics of infinitely many free fermions in an inverse harmonic potential. The ordinary vacuum of two dimensional string theory corresponds to the Fermi surface $\frac{1}{2}(p^2-x^2)=E_0$. The fluctuations on this fermi surface correspond to the massless scalar field in the closed string theory (e.g., see the review [11] and references therein).

If we get rid of a band of fermions with the energy $E_2 \leq E \leq E_1$ assuming $E_1 < E_0$, then we have three Fermi surfaces at $E=E_0$, $E=E_1$ and $E=E_2$. Thus this system possesses three massless bosons φ_0, φ_1 and φ_2 as fluctuations (or collective fields) on these Fermi surfaces (see figure 1). Interestingly, the second one φ_1 turns out to be a ghost, i.e. it has the wrong sign in its kinetic term. This is because the higher energy region is completely filled at the Fermi surface $E=E_1$ and excitations always have negative energy. On the other hand, since the opposite is true at $E=E_0$ and $E=E_2$, the fields φ_0 and φ_2 are ordinary massless scalar fields.

Despite its seeming instability near $E=E_1$, this system is stable because the fermions in the $c=1$ matrix model are free. To be exact, we need to consider type 0 string theory [5, 6] to obtain its non-perturbative completion. In the limit $E_1 \to E_2$ where the band disappears, the scalar field φ_1 and the ghost field φ_2 cancel out as expected. In this way we find a physically sensible theory with ghosts. Then we can apply the modern identification of the fermions with D0-branes in two dimensional string theory [4, 6, 12] to find what background in two dimensional string this configuration corresponds to.

[1] A closely related remark was also made in [6], where the boundary state for a hole state was considered.

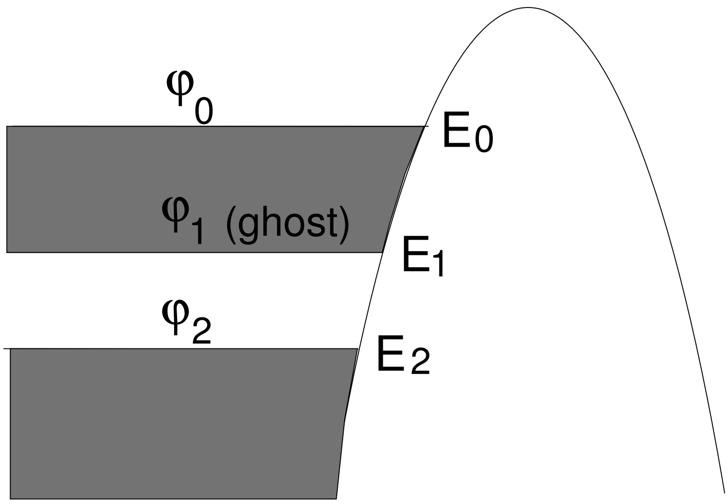

Figure 1: In the $c = 1$ matrix model, a ghost field can appear as a collective field when we remove a band of fermions.

Removing fermions with energy $E_2 \leq E \leq E_1$ can be interpreted as condensing infinitely many ghost D0-branes. Notice also that we cannot describe such a background in the two dimensional effective dilaton-gravity theory.[2]

Also closely related to ghost D-branes are the anti D-branes in topological string theory [14]. Indeed the amplitudes of the anti D-branes in topological string theory precisely cancel the amplitudes of the D-branes. The cancellation between topological branes and anti branes has been heavily used in the recent developments in topological string theory [3]. It seems that anti D-branes in topological string theory are more directly a counterpart of ghost D-branes than anti D-branes in physical superstring theory.

The paper is organized as follows. In section two, we demonstrate that the Lie superalgebras $U(N|M)$ and $OSp(N|M)$ arise in open string theory by including ghost D-branes. In section three, we discuss this open string theory and show that the branes and ghost branes cancel each other. We do this by demonstrating that theories with supergroup symmetries reduce to theories with ordinary bosonic symmetries. This part does not require the knowledge of string theory and should be readable by anyone who knows quantum field theory. Also we prove that the gauge anomaly is canceled in the type I theory. In section 4 we study the strong coupling limit of the type I $OSp(32 + 2n|2n)$ string theory and argue that it is given by the heterotic $OSp(32 + 2n|2n)$ string theory. We also discuss a supergroup extension of the heterotic $E_8 \times E_8$ string theory and mention its M-theory origin. Section 5 is devoted to the construction of type IIB-like superstrings with $U(N|N)$ gauge symmetry, motivated by S-duality in type IIB string theory. In section 6 we 'derive' Dynkin diagrams of Lie superalgebras from 7-branes configurations in type IIB string theory. In section 7 we discuss other implications of our results and future directions.

[2]A similar situation appears in the flux background of the two dimensional type 0B string theory. A description of this background was proposed in [13] using a quantum field redefinition of the RR scalar field.

2. Chan-Paton factors and Lie superalgebras

Usually, the gauge group which appears in the open string sector is $U(N)$ in type II string theory, and $SO(N)$ or $Sp(N)$ in type I string theory. These are realized by assigning Chan-Paton matrices to open strings. We will generalize these Chan-Paton matrices in superstring theories into elements of Lie superalgebras. For reviews of Lie superalgebras refer to [15 − 17] as well as appendix A of the present paper. Our results in this section, section 3 and section 6 can also be applied to bosonic and type 0 strings. Though in this paper we only consider BPS configurations of D-branes, it is straightforward to apply similar arguments to non-BPS configurations such as brane-antibrane systems [18].

2.1 $U(N|M)$ and type II string theory

We argue that this extension corresponds the inclusion of negative tension Dp-branes. We call them ghost Dp-brane. In the boundary state formalism, an ordinary BPS D-brane is represented by a boundary state $|D\rangle = |D\rangle_{\rm NSNS} + |D\rangle_{\rm RR}$. The ghost D$p$-brane is defined simply by the boundary state $|gD\rangle$ with an overall minus sign

$$|gD\rangle = -|D\rangle. \tag{2.1}$$

It is crucial to distinguish a ghost D-brane from an anti D-brane. The boundary state for the latter is obtained by flipping the sign of the RR part only [18]. A brane anti-brane system is non-supersymmetric, while a brane ghost-brane system possesses sixteen supersymmetries. This is because a ghost D-brane keeps the same half supersymmetries as the BPS D-brane does.

Consider an cylinder amplitude between a Dp-brane and a ghost Dp-brane in type II string theory. This is obviously given by minus the ordinary amplitude between BPS Dp-branes

$$\langle gD|\Delta|D\rangle = -\langle D|\Delta|D\rangle, \tag{2.2}$$

where Δ is the closed string propagator. Interpreted in the open string channel, the spectrum is given by replacing bosons in the ordinary spectrum on D-branes with fermions and vice versa. Thus all fields between a Dp-brane and a ghost Dp-brane are ghost-like. The gluons become fermions and the gauginos become bosons. Now consider N Dp-branes and M ghost Dp-branes on top of each other. We can summarize this field content by a hermitian supermatrix (see appendix A)

$$\Phi = \begin{pmatrix} \phi_1 & \psi \\ \psi^\dagger & \phi_2 \end{pmatrix}, \tag{2.3}$$

where ϕ_1 and ϕ_2 are bosonic hermitian matrices, while ψ is a complex fermionic matrix. The diagonal part ϕ_1 (or ϕ_2) corresponds to the open strings between D-branes (or ghost D-branes). The off-diagonal part ψ corresponds to the open strings between D-branes and ghost D-branes and thus they have the opposite statistics. In other words, the world-volume theory on N Dp-branes and M ghost Dp-branes is given in low energy by a $p+1$ dimensional super Yang-Mills theory with gauge group $U(N|M)$. In this way, a $U(N|M)$

JHEP03(2006)062

valued Chan-Paton factor naturally appears. If we place N D9-branes and N ghost 9-branes, the tadpoles are canceled and we obtain a consistent ten dimensional $U(N|N)$ super Yang-Mills theory coupled to type IIB supergravity.[3] Note that in this case the NSNS tadpole is also zero and therefore we do not need to invoke the Fischler-Susskind mechanism.

The appearance of the supergroup $U(N|M)$ in a similar way was mentioned in [6] in the context of matrix model duals of two dimensional string theories. In the topological string context, the gauge group of the brane-anti brane system was argued to be $U(N|M)$ in [14]. This is consistent with our discussion because in topological string theory we only have the RR-sector part [21] and thus the boundary state for an anti D-brane is precisely minus that of a D-brane.

2.2 $OSp(N|M)$ and type I string theory

To define type I string theory, we need an orientation projection. D1 or D9-branes give an $SO(N)$ Chan-Paton factor, while D5-branes[4] give $Sp(N)$ (N is even) [22]. Since there exists an orientifold 9-plane in the background, the brane configuration is described by the sum

$$|D\rangle + |\Omega\rangle, \tag{2.4}$$

where $|\Omega\rangle$ is the crosscap state. From the overlap of two of such states we get the cylinder + Möbius strip + Klein bottle amplitudes. We extend this theory by including negative tension Dp-branes as before. Notice that the $|\Omega\rangle$ part remains the same because we do not want to modify the theory itself. Then the ghost brane configuration is described by

$$-|D\rangle + |\Omega\rangle. \tag{2.5}$$

Consider the system of N D9 (or D1)-branes and $M/2$ ghost D9 (or D1)-branes, assuming M is even.[5] For open strings between two ghost D9 branes, then it is obvious from (2.5) that we get an extra minus sign for the Möbius amplitudes. This means that the Chan-Paton factor for the ghost D9 branes is in $Sp(M)$. The open strings between D9 and ghost D9 become ghost-like as before. We thus find the $OSp(N|M)$ gauge group in type I string theory. On the other hand, N D5-branes and M ghost D5-branes give rise to the gauge group $OSp(M|N)$ with an opposite sign for the coupling constant g^2_{YM}. The detailed structure of $OSp(N|M)$ will be discussed in the next subsection and in appendix A.

[3] In the presence of N D9-branes and N anti D9-branes, the system is described by the $U(N) \times U(N)$ gauge theory with bi-fundamental (complex) tachyon fields [18, 19]. Clearly there is no $U(N|N)$ symmetry. Instead, the mathematical structure of the gauge theory is described by the Quillen's superconnection, where odd elements correspond to tachyon fields (and not the gauge field) as shown in [20].

[4] Our convention for symplectic groups is such that $Sp(2) = SU(2)$. Sometimes $Sp(N)$ is denoted by $USp(N)$.

[5] The reason why we have to mod the number of ghost 9-branes by the factor 2 is that the Sp projection will be imposed on them as we will see shortly and therefore they only make sense when M is an even integer.

For 9-branes, we need to impose the condition $N - M = 32$ so that the tadpoles are canceled. As we will see later, indeed the gauge anomaly is canceled in super Yang-Mills theory with gauge group $OSp(32 + 2n|2n)$ by the Green-Schwarz mechanism [23].[6]

2.3 Ω action and Lie superalgebras

So far we have been discussing ghost D-branes from the viewpoint of boundary states. Here we consider how the orientation projection Ω acts on open string states.

The Ω action reverses the orientation of the world-sheet as usual. It always acts on the oscillator part of a massless gluon state as $\Omega\, |\text{gluon}\rangle = -|\text{gluon}\rangle$. Below we concentrate on its action on the Chan-Paton factor. Since it exchanges the two boundaries of the open string world-sheet, Ω will act like $|i,j\rangle \to |j,i\rangle$, where $i, j = 1, 2, \cdots, N + M$ in the presence of N D-branes and $M/2$ ghost-branes. We express the Chan-Paton matrix by λ and assume that λ is hermitian as usual. We expect that Ω acts by transposing λ. When we λ is a supermatrix we need to modify the definition of a transposed matrix. Indeed the effective Yang-Mills action on branes is $S \propto \mathrm{Str}F^2 + \cdots$ and the Ω action should be a symmetry of this system. Because in general $(\lambda_1\lambda_2)^T \neq \lambda_2^T\lambda_1^T$ for supermatrices, $\lambda \to \lambda^T$ is not a symmetry. As is explained in appendix A, we should supertranspose λ to $\lambda^{\tilde{T}}$. This satisfies $(\lambda_1\lambda_2)^{\tilde{T}} = \lambda_2^{\tilde{T}}\lambda_1^{\tilde{T}}$ and $\mathrm{Str}\lambda = \mathrm{Str}\lambda^{\tilde{T}}$. Notice also that $\tilde{T}$ is not a $\mathbb{Z}_2$ action since $(\lambda^{\tilde{T}})^{\tilde{T}} = K\lambda K$, where $K = \mathrm{diag}(I_N, -I_M)$. The matrix I_M denotes the $M \times M$ identity matrix.

Then we can write the Ω action in the following form

$$\Omega : \lambda \to \gamma^{-1}\lambda^{\tilde{T}}\gamma, \tag{2.6}$$

where γ is a $U(N|M)$ matrix corresponding to a gauge transformation on D-branes. By requiring that this is a $\mathbb{Z}_2$ action $\Omega^2 = 1$, we find the condition[7] for γ

$$\gamma^{\tilde{T}} = \pm\gamma\, K. \tag{2.7}$$

Under the $U \in U(N|M)$ rotation $\lambda \to U\lambda U^{-1}$, the matrix γ transforms as $\gamma \to U\gamma U^{\tilde{T}}$. We can pick up the following solution to the constraint (2.7)

$$\begin{aligned} \gamma_{OSp} &= \begin{pmatrix} I_N & 0 \\ 0 & \eta_M \end{pmatrix}, \\ \gamma_{SpO} &= \begin{pmatrix} \eta_N & 0 \\ 0 I_M & \end{pmatrix}, \end{aligned} \tag{2.8}$$

corresponding to the $\pm$ sign in (2.7), respectively. The $M \times M$ matrix η_M is defined by $\eta = -\sigma_2 \otimes I_{M/2}$ (see also (A.14) in appendix A).

The massless gauge field should satisfy $\Omega = -1$ under the action (2.6). In the case $\gamma = \gamma_{OSp}$, this leads precisely to the condition that the Chan-Paton factor is a generator of $OSp(N|M)$:

$$\lambda = -\gamma_{OSp}\, \lambda^{\tilde{T}}\, \gamma_{OSp}^{-1}. \tag{2.9}$$

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[6] With $32 + n$ D9-branes and n anti D9-branes, the gauge group becomes $SO(32 + n) \times SO(n)$ (or $Sp(32 + n) \times Sp(n)$ by considering the opposite Ω projection) [24, 25].

[7] Here we employed Schur's lemma for supermatrices [15]. Also we assumed the generic situation $N \neq M$.

This is realized when we consider ghost D9 or D1-branes in type I string theory.[8] The other case γ_{SpO} corresponds to $OSp(M|N)$ and occurs when we consider D5-branes.

3. World-volume theory on branes and ghost branes

The world-volume theory on N Dp-branes and M (or $M/2$) ghost Dp-branes can clearly be described by a gauge theory with $U(N|M)$ (or $OSp(N|M)$) Chan-Paton matrices. The low energy action[9] is that of the $(p+1)$ dimensional super Yang-Mills theory whose gauge group is one of these supergroups

$$S = \frac{1}{g_{YM}^2} \int d^{p+1}x \ \mathrm{Str} \left[-\frac{1}{4}(F_{\mu\nu})^2 - \frac{1}{2}(D_\mu \phi^i)^2 + \cdots \right], \tag{3.1}$$

where $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu - i[A_\mu, A_\nu]$ is the field-strength and ϕ^i are the transverse scalars. This action (3.1) is invariant under the gauge transformation[10] $\delta A_\mu = \partial_\mu \chi + i[\chi, A_\mu]$. Since the gauge field and the transverse scalar fields take supermatrix values, some of the bosonic modes have an extra minus sign in the kinetic terms.[11] Also there exists a fermionic gauge and scalar fields. These super Yang-Mills theories possess sixteen supersymmetries as in the ordinary case. Because of this, we can indeed extend the various string dualities consistently as we will explain later.

3.1 Cancellation between branes and ghost branes

Let us assume all branes and ghost branes are situated at the same location. In the boundary state formalism in section 2, it is clear that the branes and ghost branes cancel each other. Thus the open string theory on N Dp-branes and M ghost Dp-branes is equivalent to the theory with $N - M$ Dp-branes when $N \geq M$ or the one with $M - N$ ghost Dp-branes when $M > N$. For example, the partition function Z of the open string theory should satisfy the following equality

$$Z\left[U(N|M), g_{YM}^2\right] = Z\left[U(N-M), g_{YM}^2\right], \tag{3.2}$$

in type II string theory and

$$Z\left[OSp(N|M), g_{YM}^2\right] = Z\left[O(N-M), g_{YM}^2)\right], \tag{3.3}$$

[8] In the ghost D1-branes case, the transverse scalars are not in the adjoint representation and are subject to the the opposite projection $\Omega = 1$, i.e., they satisfy $\lambda = -\gamma_{SpO}\ \lambda^{\tilde{T}}\ \gamma_{SpO}^{-1}$.

[9] The DBI action takes the form $S = T_{\mathrm{D}p} \int d^{p+1}x\ \mathrm{Str}\sqrt{-\det(1 + 2\pi\alpha' F_{\mu\nu})}$. It has a minus sign in front of the whole DBI action for the gauge fields with wrong-sign kinetic terms.

[10] Notice that only commutators appear in these expressions for the Yang-Mills theory. Anti-commutators which are typical in Lie superalgebras (see appendix A) arise when we expand a supermatrix-valued field $\phi(x)$ by bosonic generators T^A with coefficients $\phi_A(x)$. Only ϕ_A can be Grassmann odd and in that case we find anti-commutators $T^A T^B + T^B T^A$.

[11] Refer to [26] for classical solutions of the Einstein-Maxwell theory with a minus sign in front of kinetic terms of gauge fields as appear in ghost D-branes.

JHEP03(2006)062

in type I string theory,[12] assuming $N \geq M$. When $N < M$ we clearly obtain

$$\begin{aligned} Z\left[U(N|M), g_{YM}^2\right] &= Z\left[U(M-N), -g_{YM}^2\right], \\ Z\left[OSp(N|M), g_{YM}^2\right] &= Z\left[Sp(M-N), -g_{YM}^2\right]. \end{aligned} \tag{3.4}$$

We can also derive a similar relation for the correlation functions of gauge invariant operators.

We can also imagine a situation with branes of different dimensionalities. For example, consider the system of several D3-branes coincident with N D7-branes and M ghost D7-branes. The world-volume theory of 3-branes now has a flavor symmetry $U(N|M)$ with N usual fermionic quarks and M bosonic quarks. From the brane picture it is clear that this system is equivalent to a theory with $N-M$ usual quarks.

These statements are non-trivial in open string field theory or super Yang-Mills theory (3.1). Even though the open string theory is complicated because of infinitely many fields and interactions, the essential point of this cancellation is the combinatorics of Chan-Paton matrices. In other words, it is enough to show the reduction for corresponding zero dimensional supermatrix models. This is because all the other parts, e.g., propagators, integrations of momenta and interactions, are exactly the same due to the supergroup gauge symmetry dictates that the action is constructed out of supertraces. It may appear that we need to consider multi-supermatrix models since we have many open string fields. However, as is shown in appendix B, cancellation takes place for each Wick contraction of correlation functions. Then it is obvious that the reduction holds for multi-matrix models if we prove it for one-matrix models.

This cancellation (3.2) for $U(N|M)$ matrix model has been already shown[13] perturbatively by Feynman diagrams [29, 30] and non-perturbatively by Virasoro constraints [29]. In appendix B we review these proofs and also extend it to the $OSp(N|M)$ case to show (3.3) and (3.4). The similar reduction of $U(N|M)$ was also explained for the supergroup sigma model in [31, 32] and for the topological branes [14]. In the appendix, we also show that the reduction $U(N|M) \to U(N-M)$ holds for flavor symmetry, too.

Now, it is possible to give large vevs to ϕ^is so that the D-branes are far away from the ghost D-branes. For such a configuration the cancellations like (3.2) (3.3) are not true any more. In this paper we will not get into details of the physical interpretations of these unusual modes, but only discuss briefly in the final section.

3.2 Anomaly cancellation in type I $OSp(32+2n|2n)$ string

In order to obtain a physically sensible gauge theory, we have to require all gauge anomalies to vanish. The ten dimensional $N=1$ super Yang-Mills theory always suffers from the hexagon anomaly. However, when we couple the gauge theory with $N=1$ supergravity, the anomaly is canceled by the Green-Schwarz mechanism [23].

[12] Here we consider $p = 1, 9$ branes. For $p = 5$ branes we have only to replace g_{YM}^2 with $-g_{YM}^2$ in the right-hand side of (3.3).

[13] Similar matrix models have been discussed in [27] [28] in order to compute superpotentials of 4D $N=1$ super Yang-Mills theories.

In this cancellation mechanism [23, 33], the essential identity required was

$$\mathrm{Tr}_{ad}[F^6] = \frac{1}{48}\,\mathrm{Tr}_{ad}[F^4]\cdot\mathrm{Tr}_{ad}[F^2] - \frac{1}{14400}\left(\mathrm{Tr}_{ad}[F^2]\right)^3, \tag{3.5}$$

where F is the gauge field strength. In the end, we find that (3.5) is satisfied for the celebrated gauge groups $SO(32)$ and $E_8\times E_8$.

In our case of the supergroups, we need to replace Tr_{ad} with the supertrace Str_{ad}. To see that (3.5) is satisfied in this case, we need to rewrite the supertrace Str_{ad} in the adjoint representation in terms of the supertrace Str in the fundamental representation. This can be found from the explicit form of the generators in the adjoint representation in terms of the generators $t^A_{\mu\nu}$ in the fundamental representation:

$$\begin{aligned}(t^A)^{\mu\nu}_{\rho\lambda} = \frac{1}{2}\Big[&\delta_{\lambda\nu}t^A_{\mu\rho} - (-1)^{(|\mu|+|\lambda|)(|\lambda|+|\nu|)}\delta_{\mu\rho}t^A_{\lambda\nu}\\ &+(-1)^{(|\mu|+|\rho|)(|\nu|+1)}\gamma_{\mu\lambda}\gamma_{\rho\alpha}t^A_{\alpha\nu} - (-1)^{|\lambda|(|\lambda|+|\nu|)}\gamma_{\alpha\lambda}\gamma_{\rho\nu}t^A_{\mu\alpha}\Big].\end{aligned} \tag{3.6}$$

The supertrace is defined as $\mathrm{Str}M = (-1)^{|\mu|}M_{\mu\mu}$ in the fundamental representation and as $Str_{ad}M = (-1)^{|\mu|+|\nu|}M^{\mu\nu}_{\mu\nu}$ in the adjoint representation. Using (3.6), we can see that the results for the supergroup $OSp(N|M)$ are obtained from those for $O(N-M)$ by replacing[14] traces with supertraces (for some details see appendix B.4). Explicitly, we find

$$\begin{aligned}\mathrm{Str}_{ad}[F^2] &= (N-M-2)\ \mathrm{Str}[F^2],\\ \mathrm{Str}_{ad}[F^4] &= (N-M-8)\ \mathrm{Str}[F^2] + 3\left(\mathrm{Str}[F^2]\right)^2,\\ \mathrm{Str}_{ad}[F^6] &= (N-M-32)\ \mathrm{Str}[F^6] + 15\mathrm{Str}[F^2]\mathrm{Str}[F^4].\end{aligned} \tag{3.7}$$

The condition (3.5) is satisfied only when $N-M=32$. We conclude that the anomaly is canceled for the gauge group[15] $OSp(32+2n|2n)$ as we expected.

The type IIB system with N D9-branes and N ghost D9-branes also has a gauge anomaly which gets cancelled by the Green-Schwarz mechanism.

4. Heterotic strings with supergroup gauge symmetries

4.1 Heterotic world-sheet from type I/heterotic duality

Since type I string theory with the $OSp(32+2n|2n)$ gauge group has sixteen supersymmetries, it is well-motivated to consider its strong coupling limit. We claim that it is given by the $OSp(32+2n|2n)$ heterotic string theory[16] generalizing the well-known case of $n=0$, i.e., the type I/heterotic duality [8]. The existence of this novel heterotic string has already been mentioned in [7] recently.

[14] The same is true for $SU(N|M)$ and $SU(N-M)$.

[15] It will also be interesting to check the anomaly cancellation for $E(8+\frac{n}{2},\frac{n}{2})\times E(8+\frac{n}{2},\frac{n}{2})$ that will be discussed in the next section.

[16] In this paper, we omit the difference of supergroups analogous to the familiar one between $SO(32)$ and $Spin(32)/\mathbb{Z}_2$.

JHEP03(2006)062

We can derive its world-sheet theory from the type I side. A D1-brane in the type I $OSp(32+2n|2n)$ string theory is dual to a fundamental heterotic string. We find eight bosons X^m $(m=1,2,\cdots,8)$ and eight right-moving fermions $S_R^{\tilde{a}}$ $(\tilde{a}=1,2,\cdots,8)$ as the transverse scalars and their super-partners on the brane. In addition, we find $32+2n$ left-moving fermions λ^i $(i=1,2,\cdots,32+2n)$ and $2n$ left-moving bosons $\zeta^{\tilde{\imath}}$ $(\tilde{\imath}=1,2,\cdots,2n)$ from 1-9 strings.[17] Here we used the fact that the open strings between the D1-brane and the n ghost D9-branes become ghost-like and have the wrong spin-statistics relations. The world-sheet theory of the $OSp(32+2n|2n)$ heterotic string in the light-cone Green-Schwarz formalism is thus the $N=(0,1)$ conformal field theory with field content

$$\begin{aligned}
&\text{left-moving}: \ (X_L^m,\ \lambda_L^i,\ \zeta_L^{\tilde{\imath}}), \quad m=1,2,\cdots,8;\ i=1,2,\cdots,32+2n;\ \tilde{\imath}=1,2,\cdots,2n\\
&\text{right-moving}: \ (X_R^m,\ S_R^{\tilde{a}}), \qquad m,\tilde{a}=1,2,\cdots,8.
\end{aligned} \tag{4.1}$$

The current algebra part of the heterotic world-sheet theory has $32+2n$ spin-1/2 real fermions and $2n$ spin-1/2 real bosons, which has the total central charge $c=16$.[18] Indeed, this is the free field representation of the level-one $OSp(32+2n|2n)$ current algebra [35].

4.2 Level-one $OSp(M|N)$ current algebra

We now study the level-one $OSp(M|N)$ current (N is even) algebra since it is an essential building block of the heterotic $OSp(32+2n|2n)$ string theory. For general aspects of current algebras based on supergroups refer to, e.g., [35, 36].

We have the following OPEs for the M free real fermions and N real bosons (called symplectic bosons):

$$\lambda^i(z)\lambda^j(0)\sim\frac{\delta^{ij}}{z}, \qquad \zeta^{\tilde{\imath}}(z)\zeta^{\tilde{\jmath}}(0)\sim\frac{-\eta^{\tilde{\imath}\tilde{\jmath}}}{z}. \tag{4.2}$$

We will raise the index of symplectic bosons as $\zeta^{\tilde{\imath}}=-\eta^{\tilde{\imath}\tilde{\jmath}}\zeta_{\tilde{\jmath}}$, where the anti-symmetric matrix $\eta^{\tilde{\imath}\tilde{\jmath}}=iJ^{\tilde{\imath}\tilde{\jmath}}$ is defined by (A.14) in appendix A. We define the bosonic currents

$$J^a=\frac{1}{2}t^a_{ij}\lambda^i\lambda^j, \quad J^{\tilde{a}}=\frac{1}{2}t^{\tilde{a}\tilde{k}}_{\tilde{\imath}}\zeta^{\tilde{\imath}}\zeta_{\tilde{k}}, \tag{4.3}$$

The bosonic generators t^a and $t^{\tilde{a}}$ belong to the $SO(M)$ and $Sp(N)$ Lie algebras, respectively:

$$t^a_{ij}=t^a_{ji}, \quad t^{\tilde{a}i}_{j}=-\eta^{ik}t^{\tilde{a}l}_{k}\eta_{lj}. \tag{4.4}$$

We also require that t^a and $t^{\tilde{a}}$ are hermitian. In addition, there exist fermionic currents[19]

$$J^\alpha=t^\alpha_{\tilde{\imath}j}\zeta^{\tilde{\imath}}\lambda^j. \tag{4.5}$$

[17] If we consider more than one D-strings, there are gauge fields and their anomaly must be canceled [34]. The gauge anomaly is indeed canceled in our configuration of 9-branes.

[18] As we check in appendix C, the central charge of the level-one current algebra for $OSp(M|N)$ is $c=(N-M)/2$. Thus when $N-M=32$ the total central charge vanishes in the left-moving sector: $c_L=10+16-26=0$.

[19] Notice that the matrix t^α is bosonic.

Then it is easy to show the currents (4.3) and (4.5) give a representation of the $OSp(M|N)$ current algebra at level-one

$$J^A(z)J^B(0) \sim \frac{\frac{1}{2}\mathrm{Str}[t^A t^B]}{z^2} + \frac{if^{AB}{}_C\, J^C(0)}{z}, \tag{4.6}$$

where A denotes all indices for the adjoint representations of $OSp(32+2n|2n)$, i.e. $A = \{a, \tilde{a}, \alpha\}$. The structure constants $f^{AB}{}_C$ are defined by $[t^A, t^B] = if^{AB}{}_C\, t^C$, where $[,]$ denotes an anti-commutator if A and B are fermionic; otherwise it denotes a commutator (see appendix A).

4.3 Closed string spectrum in the heterotic $OSp(32+2n|2n)$ string

In order to keep modular invariance, we impose the GSO projection for the (λ, ζ) system in a way similar to the $SO(32)$ heterotic string [37, 33]. In the NS sector we have anti-periodic boundary conditions $\lambda^i(\tau, \sigma+2\pi) = -\lambda^i(\tau,\sigma)$ and $\zeta^{\tilde{\imath}}(\tau,\sigma+2\pi) = -\zeta^{\tilde{\imath}}(\tau,\sigma)$. The R sector is defined by periodic boundary conditions. Let us define F_λ and F_ζ modulo 2 to be the operators that count the numbers of λ and ζ relative to the appropriate ground state in each sector.[20] The GSO projection picks out states with $(-1)^{F_\lambda+F_\zeta} = 1$ in both sectors. We define the torus partition function by

$$Z = \mathrm{Tr}\frac{1+(-1)^{F_\lambda+F_\zeta}}{2}(-1)^{F_\zeta} = \mathrm{Tr}\frac{(-1)^{F_\lambda}+(-1)^{F_\zeta}}{2}, \tag{4.7}$$

where the trace is over the left-moving NS and R sectors. The latter expression can be written as a sum over the four spin structures of the torus, and reduces to the torus partition function of the heterotic $SO(32)$ string due to cancellation between $2n$ fermions and $2n$ bosons. Note that the cancellation requires the same periodicities of bosons and fermions.[21] These are the reasons for the unconventional definition above and then the modular invariance is obvious. The insertion of $(-1)^{F_\zeta}$ in the intermediate expression implies that states with an odd number of ζ excitations have the opposite statistics to the usual heterotic states. The total zero-point energies[22] of these sectors are given by -1 and $+1$ in the NS and R sectors. Due to level matching with the right-moving sector, there is no tachyon. The graviton multiplet arises in the standard way. Massless gauge fields only come from the NS sector of the current algebra. We find states corresponding to the $SO(32+2n)\times Sp(2n)$ gauge bosons

$$\lambda^i_{-1/2}\lambda^j_{-1/2}|0\rangle, \quad \zeta^{\tilde{\imath}}_{-1/2}\zeta^{\tilde{\jmath}}_{-1/2}|0\rangle, \tag{4.8}$$

as well as their gauginos. Furthermore there exist gauge fields (and bosonic gauginos)

$$\lambda^i_{-1/2}\zeta^{\tilde{\jmath}}_{-1/2}|0\rangle, \tag{4.9}$$

[20] This ground state is the $SL(2,\mathbb{C})$-invariant ground state in the NS sector, and $|0\rangle_{\mathrm{R}}$ that will be defined later in the R sector.

[21] This also follows from the $\mathbb{Z}_2$ residual gauge symmetry on the D-string.

[22] The physical state condition is $(L_0+a)|\mathrm{phys}\rangle = 0$, where a is the zero-point energy.

JHEP03(2006)062

which are fermionic because $(-1)^{F_\zeta} = -1$. Altogether, they form the ten dimensional $OSp(32+2n|2n)$ super Yang-Mills multiplet.

Recently, it was found in [38] that there exists an open string in the $SO(32)$ heterotic string theory. One important consistency check for the existence of such an open string was the cancellation of gauge non-invariant terms between the world-sheet and spacetime. Another was the conservation of degrees of freedom flowing from the world-sheet to spacetime. These checks go through for the $OSp(32+2n|2n)$ heterotic string by replacing the traces by supertraces as we did in subsection 3.2.

Another way to describe this heterotic string theory is to bosonize the (λ, ζ) system. We can regard the n copies of the symplectic bosons $(\zeta^{2l-1}, \zeta^{2l})$ $(l = 1, 2, .., n)$ as the (β, γ) systems. Then we can apply the standard bosonization procedure $(k = 1, 2, ..., 16+n)$

$$\begin{aligned} \lambda^k &\equiv \frac{1}{\sqrt{2}}(\lambda^{2k-1} + i\lambda^{2k}) = e^{i\varphi^k}, \qquad \bar{\lambda}^k \equiv \frac{1}{\sqrt{2}}(\lambda^{2k-1} - i\lambda^{2k}) = e^{-i\varphi^k}, \\ \zeta^l &\equiv \frac{1}{\sqrt{2}}(\zeta^{2l-1} + i\zeta^{2l}) = e^{i\tilde{\varphi}^l}\partial\xi^l, \qquad \bar{\zeta}^l \equiv \frac{1}{\sqrt{2}}(\zeta^{2l-1} - i\zeta^{2l}) = e^{-i\tilde{\varphi}^l}\eta^l, \end{aligned} \tag{4.10}$$

where the OPEs are

$$\varphi^k(z)\varphi^{k'}(0) \sim -\delta^{kk'}\log z, \quad \tilde{\varphi}^l(z)\tilde{\varphi}^{l'}(0) \sim \delta^{ll'}\log z, \quad \eta^l(z)\xi^{l'}(0) \sim \frac{\delta^{ll'}}{z}. \tag{4.11}$$

We can represent the Cartan subalgebra generators H_m $(m = 1, 2, \cdots, 16+2n)$ of $OSp(32+2n|2n)$ as the following $16+2n$ currents

$$\begin{aligned} H_l &= \zeta^l\bar{\zeta}^l = -i\partial\tilde{\varphi}^l, \\ H_{n+k} &= \lambda^k\bar{\lambda}^k = i\partial\varphi^k. \end{aligned} \tag{4.12}$$

The vertex operator of the form ($V_{\mathrm{oscillator}}$ denotes the part made of oscillators of $\varphi, \tilde{\varphi}, \eta$ and ξ)

$$V_{\vec{\alpha}} = \exp\left[i\sum_{k=1}^{16+n}\alpha_{n+k}\varphi^k + i\sum_{l=1}^{n}\alpha_l\tilde{\varphi}^l\right]\cdot V_{\mathrm{oscillator}}, \tag{4.13}$$

possesses the weight eigenvalues $H_m = \alpha_m$ and its conformal dimension is

$$\Delta = -\frac{1}{2}\sum_{m=1}^{n}(\alpha_m)^2 + \frac{1}{2}\sum_{m=n+1}^{16+2n}(\alpha_m)^2 + \Delta_{\mathrm{oscillator}} \equiv \frac{1}{2}(\alpha,\alpha) + \Delta_{\mathrm{oscillator}}. \tag{4.14}$$

Notice that the conformal dimension $\Delta_{\mathrm{oscillator}}$ for the oscillator part is always a non-negative integer.

The $16+2n$ simple roots of $OSp(32+2n|2n)$ (see the Dynkin diagram in figure 6) are given by the following operators (we omit cocycle factors)

$$\begin{aligned} J^{\alpha_l} &= e^{i\tilde{\varphi}^l - i\tilde{\varphi}^{l+1}}\partial\xi^l\eta^{l+1} \qquad (l = 1, 2, \cdots, n-1), \\ J^{\alpha_n} &= e^{i\tilde{\varphi}^n - i\varphi^1}\partial\xi^n, \\ J^{\alpha_{n+k}} &= e^{i\varphi^k - i\varphi^{k+1}} \qquad (k = 1, 2, \cdots, 15+n), \\ J^{\alpha_{16+2n}} &= e^{i\varphi^{15+n} + i\varphi^{16+n}}. \end{aligned} \tag{4.15}$$

Notice that α_n is the only fermionic simple root and the others are all bosonic. We find the inner products (called the symmetric Cartan matrix[23] $A'_{mm'}$)

$$\begin{aligned}
(\alpha_m, \alpha_{m'}) &= -2 \quad \text{if } 1 \leq m = m' \leq n-1, \\
&= 2 \quad \text{if } n+1 \leq m = m' \leq 16+2n, \\
&= 1 \quad \text{if } \alpha_m \text{ and } \alpha_{m'} \text{ are adjacent in the Dynkin diagram and} 1 \leq m, m' \leq n, \\
&= -1 \quad \text{if } \alpha_m \text{ and } \alpha_{m'} \text{ are adjacent in the Dynkin diagram and} \\
&\quad\ n \leq m, m' \leq 16+2n, \\
&= 0 \quad \text{in other cases.}
\end{aligned} \tag{4.16}$$

From (4.15), we see that the root lattice of $OSp(32+2n|2n)$ is given by

$$\Gamma_{\text{root}} = \{(n_1, ..., n_{16+2n}) \in \mathbb{Z}^{16+2n} | \textstyle\sum_{l=1}^{n} n_l + \sum_{k=1}^{16+n} n_{n+k} \in 2\mathbb{Z}\} \tag{4.17}$$

The weights of the 'spinor' representation can be found as follows. In terms of the $\lambda\zeta$ system, the fields corresponding to simple roots are

$$\begin{aligned}
J^{\alpha_l} &\propto \bar{\zeta}^l \zeta^l \qquad (l = 1, 2, \cdots, n-1), \\
J^{\alpha_n} &\propto \bar{\lambda}^1 \zeta^{2n-1}, \\
J^{\alpha_{n+k}} &\propto \bar{\lambda}^{k+1} \lambda^k \qquad (k = 1, 2, \cdots, 15+n), \\
J^{\alpha_{16+2n}} &\propto \lambda^{15+n} \lambda^{16+n}.
\end{aligned} \tag{4.18}$$

As in the $SO(32)$ case, the GSO projected R-ground states furnish a 'spinor' representation. Let us define $|0\rangle_R$ to be the ground state annihilated by the zero-modes of λ^k and ζ^l. Since the simple root operators contain at least one of these, $|0\rangle_R$ is the highest weight state in one of the irreducible spinor representations. This state has eigenvalues (highest weight)

$$((-1/2)^n, (1/2)^{16+n}) \tag{4.19}$$

of the Cartan generators. Thus an element of the weight lattice Γ_{spinor} for this spinor representation is the sum of (4.19) and a vector in Γ_{root}. The Narain lattice for the bosonized description of the $OSp(32+2n|2n)$ heterotic string is the sum of the two lattices:

$$\Gamma_{n,16+n} = \Gamma_{\text{root}} \cup \Gamma_{\text{spinor}}. \tag{4.20}$$

The inner product of $q, q' \in \Gamma_{n,16+n}$ is

$$(q, q') = -\sum_{l=1}^{n} q_l q'_l + \sum_{k=1}^{16+n} q_{n+k} q'_{n+k}. \tag{4.21}$$

as we defined in (4.14). It is easy to check that the lattice $\Gamma_{n,16+n}$ is even with respect to this inner product as required by the level matching condition and locality of vertex operators. Although the torus partition function has contributions from fermions η, ξ, we expect that modular invariance requires self-duality of the lattice. We have checked that the lattice $\Gamma_{n,16+n}$ is indeed self-dual.

[23]Distinguish this from the ordinary Cartan matrix $A_{mm'}$ whose diagonal part is always 2 or 0.

4.4 Heterotic string based on the $E_8 \times E_8$-like supergroup

We can define another heterotic string by imposing double GSO projections as we do to define the $E_8 \times E_8$ heterotic string [37] (see also [7] for an earlier discussion). We expect that this leads to another supergroup extension of heterotic string theory in ten dimensions. Let n be even. We divide the free fields λ^i and $\zeta^{\tilde{\imath}}$ in the previous subsection into two groups:

$$\begin{aligned} (\lambda^i, \zeta^{\tilde{\imath}}): &\quad i = 1, 2, \cdots, 16+n; \quad \tilde{\imath} = 1, 2, \cdots, n, \\ (\lambda^{i'}, \zeta^{\tilde{\imath}'}): &\quad i' = 17+n, 18+n, \cdots, 32+2n; \quad \tilde{\imath}' = n+1, n+2, \cdots, 2n. \end{aligned} \tag{4.22}$$

After taking GSO projections (4.7) separately on these theories, we have the four left-moving sectors (NS, NS), (NS, R), (R, NS) and (R, R). As in the OSp case, the (NS, NS) sector has the total zero-point energy -1 and thus the massless gauge bosons for $SO(16+n) \times SO(16+n) \times Sp(n) \times Sp(n)$ are

$$\lambda^i_{-1/2}\lambda^j_{-1/2}|0\rangle, \quad \lambda^{i'}_{-1/2}\lambda^{j'}_{-1/2}|0\rangle, \quad \zeta^{\tilde{\imath}}_{-1/2}\zeta^{\tilde{\jmath}}_{-1/2}|0\rangle, \quad \zeta^{\tilde{\imath}'}_{-1/2}\zeta^{\tilde{\jmath}'}_{-1/2}|0\rangle. \tag{4.23}$$

There also exist fermionic gauge fields

$$\lambda^i_{-1/2}\zeta^{\tilde{\jmath}}_{-1/2}|0\rangle, \quad \lambda^{i'}_{-1/2}\zeta^{\tilde{\jmath}'}_{-1/2}|0\rangle. \tag{4.24}$$

Furthermore, we have other massless states from the (NS, R) and (R, NS) sectors. Since the zero-point energy vanishes, the ground states give rise to massless fields. The degeneracy of ground states comes from the fermionic zero-modes $(\lambda^i_0, \lambda^{i'}_0)$ and bosonic zero-modes $(\zeta^{\tilde{\imath}}_0, \zeta^{\tilde{\imath}'}_0)$. The former, as is familiar in the ordinary heterotic string theory, leads to the spinor representations with dimension $2^{7+\frac{n}{2}}$ (assuming n is even) of either of the two $SO(16+n)$s. The latter is a novel ingredient in this kind of heterotic string. Indeed it generates infinitely many massless modes because the bosonic zero-modes constitute the Heisenberg algebra (or equivalently the spinor representation of the metaplectic group).

In the ordinary $E_8 \times E_8$ heterotic string, the 248 dimensional adjoint representation of one E_8 is obtained by combining the 120 dimensional adjoint representation of one $SO(16)$ in the (NS, NS) sector and the 2^7 =128 dimensional spinor representation of the same $SO(16)$. In our heterotic string we can see that the massless gauge bosons and fermions belong to two copies of an infinite dimensional Lie superalgebra. We call this superalgebra $E(8+\frac{n}{2}, \frac{n}{2})$ since it includes the E_8 algebra. What we have discussed is the heterotic $E(8+\frac{n}{2}, \frac{n}{2}) \times E(8+\frac{n}{2}, \frac{n}{2})$ string.

We found that $E(8+\frac{n}{2}, \frac{n}{2})$ is infinite dimensional. This fact seems to be consistent with the known mathematical fact: there is no finite dimensional Lie superalgebra which is a counterpart for the E_n Lie algebra. The only examples of exceptional Lie superalgebras are called $G(3)$ and $F(4)$, whose bosonic parts are $G_2 \times SU(2)$ and $SO(7) \times SU(2)$, respectively. Indeed, if we try to extend the E_8 algebra by adding fermionic roots, we find that the Cartan matrix ceases to be positive definite. Thus the superalgebra becomes infinite dimensional (this is called an indefinite superalgebra).

The Narain lattice for the $E(8+\frac{n}{2}, \frac{n}{2}) \times E(8+\frac{n}{2}, \frac{n}{2})$ heterotic string is $\Gamma_{\frac{n}{2}, 8+\frac{n}{2}} \times \Gamma_{\frac{n}{2}, 8+\frac{n}{2}}$, where $\Gamma_{\frac{n}{2}, 8+\frac{n}{2}}$ is the root lattice of $E(8+\frac{n}{2}, \frac{n}{2})$ and is defined in the same way as $\Gamma_{n, 16+n}$: An element of $\Gamma_{\frac{n}{2},8+\frac{n}{2}}$ is an integer vector $(n_1, ..., n_{8+n})$ such that $\sum_a n_a$ is even, or the sum of such a vector and $((-1/2)^{n/2}, (1/2)^{8+n/2})$. It is equipped with the $(-^{n/2}, +^{8+n/2})$-signature metric.

We can discuss the strong coupling limit of this heterotic string theory. The product form of the gauge supergroup suggests the Horava-Witten type duality [9]. In other words, we expect that this ten dimensional string theory is dual to the M-theory on $S^1/\mathbb{Z}_2$. This $\mathbb{Z}_2$ projection preserves sixteen supersymmetries. Each of two fixed planes will provide the $E(8+\frac{n}{2},\frac{n}{2})$ gauge theory. It would be interesting to see this from the anomaly cancellation argument in eleven dimensional supergravity.

Finally we would like to mention a subtlety that appears when we consider the spinor representation for the OSp supergroups. Such a state appears in the massless states of the heterotic $E(8+\frac{n}{2},\frac{n}{2})\times E(8+\frac{n}{2},\frac{n}{2})$ string as we have seen, and also in the massive states of the heterotic $OSp(32+2n|2n)$ string. The corresponding vetex operators can be constructed via the bosonization (4.10) of the (β,γ) system. To maintain the modular invariance, we need to pick up a state with a definite picture as in the superconformal ghosts sector of the ordinary superstrings. When we consider an OPE between two different R-sector operators we encounter an vertex operator with a different picture. Then we need to identify states with different pictures as we do by the picture changing operation in ordinary superstrings. We leave the details of this operation in our case for a future problem.

4.5 Toroidal compactification

One can consider the heterotic $OSp(32+2n|2n)$ or $E(8+\frac{n}{2},\frac{n}{2})\times E(8+\frac{n}{2},\frac{n}{2})$ string theory compactified on T^d. As in the usual heterotic string theory, it is convenient to use the bosonized description. We use coordinates such that the radius of each circle is R.

Turn on constant metric $g_{\mu\nu}$ $(\mu,\nu = 1, ..., d)$, B-field $B_{\mu\nu}$, and Wilson lines A^l_μ $(l = 1, ..., n)$ and A^k_μ $(k = n+1, 2, ..., 16+2n)$. As is shown in appendix D, the momenta for this system are given by

$$\begin{aligned}
k_{L\mu} &= \frac{n_\mu}{R} + \frac{w^\nu R}{\alpha'}(g_{\mu\nu}+B_{\mu\nu}) - q_l A^l_\mu - q_k A^k_\mu - \frac{w^\nu R}{2}(-A^l_\nu A^l_\mu + A^k_\nu A^k_\mu),\\
k_l &= (q_l - w^\mu R A^l_\mu)\sqrt{\frac{2}{\alpha'}},\\
k_k &= (q_k + w^\mu R A^k_\mu)\sqrt{\frac{2}{\alpha'}},\\
k_{R\mu} &= \frac{n_\mu}{R} + \frac{w^\nu R}{\alpha'}(-g_{\mu\nu}+B_{\mu\nu}) - q_l A^l_\mu - q_k A^k_\mu - \frac{w^\nu R}{2}(-A^l_\nu A^l_\mu + A^k_\nu A^k_\mu), \qquad (4.25)
\end{aligned}$$

generalizing the results for ordinary heterotic strings [22]. Here (q_l, q_k) is a point in the lattice $\Gamma_{n,16+n}$ or $\Gamma_{\frac{n}{2},8+\frac{n}{2}}\times\Gamma_{\frac{n}{2},8+\frac{n}{2}}$ that defines the heterotic string theory. n_μ and w^μ are arbitrary integers representing the momentum and the winding number along the x^μ direction.

Let us define

$$l := \left(\sqrt{\frac{\alpha'}{2}}k_{L\mu}, \sqrt{\frac{\alpha'}{2}}k_l, \sqrt{\frac{\alpha'}{2}}k_k, \sqrt{\frac{\alpha'}{2}}k_{R\mu}\right).$$

Then the level matching condition is

$$0 = L_0 - \bar{L}_0 = \frac{1}{2} l \circ l + N - \bar{N} - 1.$$

Here N and $\bar{N}$ arise from oscillator excitations and take integer values. We have defined the metric on the momentum lattice by

$$\begin{aligned} l \circ l' &= \frac{\alpha'}{2}(g^{\mu\nu} k_{L\mu} k'_{L\nu} - k_l k'{}_l + k_k k'{}_k - g^{\mu\nu} k_{R\mu} k'_{R\nu}) \\ &= n_\mu w'^\mu + w^\mu n'_\mu - q_l q'_l + q_k q'_k. \end{aligned}$$

We see that the level matching condition is satisfied because the lattice $\Gamma_{n,16+n}$ or $\Gamma_{\frac{n}{2},8+\frac{n}{2}}$ of (q_l, q_k) is even. It is clear that the moduli space of the lattices for this toroidal compactification is given by

$$SO(16+d+n, d+n; \mathbb{Z}) \Big\backslash SO(16+d+n, d+n; \mathbf{R}) \Big/ SO(16+d+n, n; \mathbf{R}) \times SO(d, \mathbf{R}). \quad (4.26)$$

The moduli are the metric, B-field, and Wilson lines.

As is well-known the $SO(32)$ and $E_8 \times E_8$ heterotic strings become equivalent upon S^1 compactification by choosing appropriate Wilson lines and radii. This equivalence extends to the $OSp(32+2n|2n)$ and $E(8+\frac{n}{2}, \frac{n}{2}) \times E(8+\frac{n}{2}, \frac{n}{2})$ strings. Let us turn on Wilson lines

$$(A^l, A^k) = ((1/2R)^{n/2}, 0^{n/2}, (1/2R)^{8+n/2}, 0^{8+n/2})$$

on the OSp side and

$$(A^l, A^k) = (0^n, 1/R, 0^{7+n/2}, 1/R, 0^{7+n/2})$$

on the $E \times E$ side. It is cumbersome but straightforward to show that the spectrum (4.25) on the OSp side is exchanged with that on the $E \times E$ side via $R \to \alpha'/2R$, $(k_L, k_R) \to (k_L, -k_R)$.

5. Type II-like closed superstrings with $U(n|n)$ supergroup gauge symmetries

In the previous section, we constructed the world-sheet of the heterotic $OSp(32+2n|2n)$ string from the strong coupling limit of a D1-brane in the type I $OSp(32+2n|2n)$ string, generalizing the typeI/heterotic duality. In this section, we consider the type IIB S-duality in the same spirit and ask what the world-sheet description is for the S-dual of the system involving n D9-branes and n ghost D9-branes. In other words we study type IIB string with n NS9-brane and n ghost NS9-branes. We will be able to construct a IIB-like superstring world-sheet which leads to the $U(n|n)$ gauge symmetry.

5.1 Superstring world-sheet from S-duality

Consider the world-sheet of a D-string in the background of n D9-branes and n ghost D9-branes. We find eight transverse scalars X^m ($m = 1, 2, \cdots, 8$), a non-dynamical gauge field A_μ, eight left-moving fermions S_L^a ($a = 1, 2, \cdots, 8$), and eight right-moving fermions $S_R^{\tilde{a}}$ ($\tilde{a} = 1, 2, \cdots, 8$) from the 1-1 string. S_L^a and $S_R^{\tilde{a}}$ transform in the spinor representations of opposite chiralities.[24] These fields from the 1-1 string match the massless excitations of the type IIB fundamental string [39]. We have new ingredients due to the 9-branes. From the 1-9 strings, we find left-moving fermions λ^i ($i = 1, ..., n$) and left-moving (ghost) bosons $\zeta^{\tilde{\imath}}$ ($\tilde{\imath} = 1, ..., n$). The 9-1 strings give the conjugate fields $\bar{\lambda}_i$ and $\bar{\zeta}_{\tilde{\imath}}$. These spin-$\frac{1}{2}$ fermions and bosons behave just like those in (4.10). They furnish a representation of the level-one current algebra $OSp(2n|2n)_{k=1}$, which will reduce to $U(n|n)_{k=1}$ as we will see below.

We also need to take into account the effects of the gauge field A_μ on the D1-brane. Let us consider a compactification on a circle of radius 1 in appropriate coordinates and assume that the D-string is wrapping the circle. It is well-known that the flux $F_{\tau\sigma}$ measures the fundamental string charge on the D-string. Since we are interested in the pure D-string (e.g. the perturbative F-string in the dual side), it is natural to consider the sector with $F_{\tau\sigma} = 0$. Then the path-integral for A_μ after gauge fixing is over the constant Wilson line A_σ. Since A_μ couples with the $U(1)$ current

$$J = -\bar{\lambda}_i \lambda^i - \bar{\zeta}_{\tilde{\imath}} \zeta^{\tilde{\imath}}, \tag{5.1}$$

the integration over constant A_σ forces the $U(1)$ charge J_0 to vanish:

$$J_0 \equiv \int d\sigma J(\sigma) = 0. \tag{5.2}$$

The restriction to the zero-charge sector reduces the current algebra[25] to $U(n|n)_{k=1}$ (see [36] for the properties of this current algebra).

Let us combine the fields as $(\Lambda^I) = (\lambda^i, \zeta^{\tilde{\imath}}), (\bar{\Lambda}_I) = (\bar{\lambda}_i, \bar{\zeta}_{\tilde{\imath}})$. Turning on a Wilson line along the S^1 changes the periodicities to $\Lambda^I(\sigma + 2\pi) = e^{2\pi i\nu}\Lambda^I(\sigma)$, $\bar{\Lambda}_I(\sigma + 2\pi) = e^{-2\pi i\nu}\bar{\Lambda}_I(\sigma)$ for some real ν. Just as the $\mathbb{Z}_2$ gauge symmetry produces the R- and NS-sectors of the heterotic string in the type I/heterotic duality, the $U(1)$ gauge symmetry instructs us to integrate over ν from 0 to 1. This integration is however trivial because as long as we look at sectors where the $U(1)$ charge J_0 vanishes,[26] the Hilbert space is independent of ν[27]. We choose to work in the NS-sector ($\nu = 1/2$) in what follows.

---

[24] Here we imposed the Dirac equation.

[25] If we fully gauge the $U(1)$ without imposing $F_{\tau\sigma} = 0$, we get the current algebra $PSU(n|n)_{k=1}$.

[26] The requirement $J_0|\text{state}\rangle = 0$ can also be regarded as the analog of the GSO projection in the heterotic string case.

[27] For example, the J_0 condition excludes $\bar{\lambda}_{i0}\bar{\lambda}_{j0}|0\rangle_{\rm R}$ that has no counterpart in the NS sector.

JHEP03(2006)062

5.2 Closed superstrings with $U(n|n)$ gauge symmetries

We are led to consider a superstring whose world-sheet theory is described in the NSR-formulation by

$$\begin{aligned} \text{left-moving} &: (X_L^\mu, \psi_L^\mu) \times U(n|n)_{k=1}, \\ \text{right-moving} &: (X_R^\mu, \psi_R^\mu), \end{aligned} \tag{5.3}$$

with the $U(1)$ projection (5.2). As is usual in the type II string, we also have the (b, c) and (β, γ) ghosts.

Now we briefly study the new superstring we have just discovered. In the $(0, 0)$-picture, the gauge fields are represented by the heterotic-like vertex operators

$$J^\alpha(z)\bar{\partial}X^\mu(\bar{z})e^{ik\cdot X} + \cdots, \tag{5.4}$$

where $J^\alpha = \bar{\Lambda}_I(t^\alpha)^I{}_J\Lambda^J$ are the $U(n|n)_{k=1}$ currents. The modular invariance again follows from the modular invariance of the usual type IIB superstring due to the cancellation between λs and ζs. Note that the partition function involves integration over the periodicity ν in the σ direction as well as the periodicity ν' in the τ-direction. The ν' integral is equivalent to the J_0 condition. Both integrals are trivial because the integrand is one.

Clearly, this superstring with $U(n|n)$ gauge symmetry is equivalent to the ordinary type IIB superstring as long as the amplitude involves only the external particles that are present in the usual theory. This is because the system with n D-branes and n ghost D-branes is equivalent to the one without branes, and our theory is S-dual to such a system. This equivalence holds perturbatively because of the fact that a $U(n|n)$ gauge theory is trivial as we saw in subsection 3.1. This equivalence is no longer true when we compactify the string theory on a circle and turn on generic Wilson lines. For example, it is possible to turn on the Wilson lines so that the interactions between the NS9-branes and the ghost NS9-branes, which carry $U(n)$ gauge groups, become weak. This may be a useful model to investigate NS9-branes.

The existence of a gauge multiplet implies that the supersymmetry is superficially broken from 32 supercharges to 16 supercharges, though the system sitting at the vacuum is actually equivalent to the type IIB superstring. We expect that the construction here extends to IIA-like superstrings via T-duality. We would like to come back to more details of these new string theories in another publication.

6. Lie superalgebras from 7-brane configurations

Type I $SO(32)$ string theory on T^2 is equivalent to the type IIB string theory on $T^2/\mathbb{Z}_2$ with four orientifold 7-branes (O7-branes) located at each fixed point via the T-duality. There are sixteen D7-branes allowed so that the tadpoles are canceled. This string theory is known to be non-perturbatively described by F-theory compactified on a specific elliptically fibered K3 surface [40, 41]. In the latter description the presence of D7-branes and O7-branes is equivalent to the existence of singular fibers in the K3 surface.

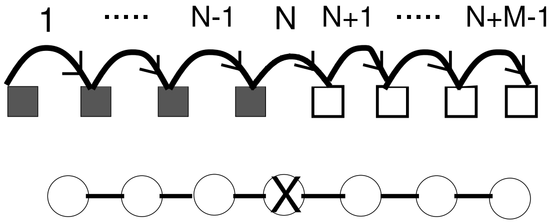

Figure 2: The $SU(M|N)$ Dynkin diagram from a 7-brane configuration in type IIB string theory theory. A white box denotes a D7-brane and a black one a ghost D7-brane. A line with an arrow between 7-branes represents an F-string, which corresponds to one of the simple roots $\alpha_1, \alpha_2, \cdot\cdot\cdot, \alpha_{N+M-1}$. The N-th F-string becomes fermionic because it stretches between a 7-brane and a ghost 7-brane agreeing with the fact that α_N is a fermionic simple root.

If we consider a probe D3-brane near an O7-brane, its low energy theory is given by the 4D $N = 2$ $SU(2)$ super Yang-Mills theory with various flavor symmetries depending on the configurations of D7-branes. For example, if we have N_f D7-branes located at the orientifold, then the flavor symmetry is $D_{N_f} = SO(2N_f)$. The quarks are realized as the (p, q) strings or their string junctions between the D3-brane and various 7-branes. It is even possible to realize the $E_{6,7,8}$ flavor symmetries. In summary, we can realize A_n, D_n and $E_{6,7,8}$ (i.e. all simply laced Lie algebras) symmetries by this method (refer to [10] and references therein). These models of 7-branes and string junctions provide us with a visual way of understanding various enhanced symmetries in string theory.

In this setup of 7-branes in type IIB string, we can again introduce ghost D7-branes without breaking the sixteen supersymmetries. Notice that a ghost 7-brane possesses the opposite monodromy $\tau(z) \sim -\frac{1}{2\pi i}\log z$ of the dilaton-axion relative to the ordinary 7-brane. Let us start with M D7-branes and N ghost D7-branes. Then we can see that the open fundamental strings between them lead to the adjoint representations of the Lie superalgebra $SU(M|N)$. Indeed we can pick up simple roots naturally from the brane configuration and find the structure of the Dynkin diagram as in Fig.2. One important new ingredient is that in the superalgebra one of the simple roots is fermionic and indeed it is realized as an open string between a D7-brane and a ghost D7-brane.

Furthermore we can add an O7-brane in this 7-brane configuration. Then the gauge group enhances into $OSp(2M|2N)$. The O7-brane can be regarded as a bound state of one $(1, -1)$ and one $(1, 1)$ 7-brane [10]. Then we can express the simple roots and the Dynkin diagram of $OSp(2M|2N)$ in terms of string junctions as in Fig.3. Even though the $SO(2M)$ part of the bosonic subgroup is manifest in Fig.3, the $Sp(2N)$ symmetry is not obvious. In fact, we can find the string junction corresponding to the long root with length-squared four which is typical in $Sp(2N)$ as in Fig.4. In terms of simple roots of the OSp algebra the long root $\alpha_{\rm long}$ is given by

$$\alpha_{\rm long} = 2(\alpha_N + \alpha_{N+1} + \cdots + \alpha_{N+M-2}) + \alpha_{N+M-1} + \alpha_{N+M}. \tag{6.1}$$

Then one may worry that the long root may contradict with the standard BPS condition $Q^2 = -(\alpha, \alpha) \geq -2$, where Q^2 is the intersection number of the string junction. In fact,

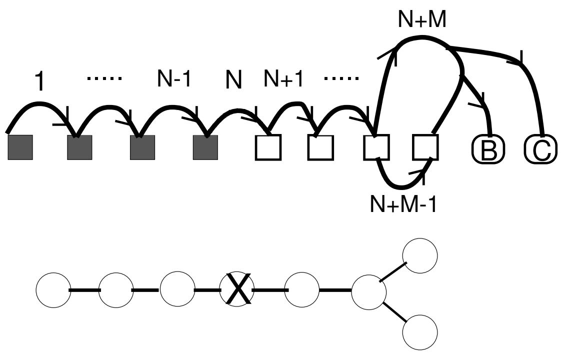

Figure 3: The $OSp(2M|2N)$ Dynkin diagram from a 7-brane configuration in type IIB string. The white box denotes a D7-brane (or $(1,0)$ 7-brane) and the black one a ghost D7-brane. The circles B and C are $(1,1)$ and $(1,-1)$ 7-branes, respectively. The N-th F-string becomes fermionic because it stretches between a 7-brane and a ghost 7-brane, corresponding to the fermionic simple root α_N. The final root α_{N+M} is the string junction made of $(2,0)$, $(1,1)$ and $(1,-1)$ strings.

this was the reason why we can not realize the non-simply laced symmetry[28] in the F-theory on K3.

However, in our model with ghost D7-branes, the long root is actually BPS. First of all, the intersection number should be defined with a minus sign for the open string which starts or ends at ghost branes. Thus we should have $(\alpha_m, \alpha_m) = -2$ for $m = 1, 2, \cdots, N-1$, $(\alpha_N, \alpha_N) = 0$, and $(\alpha_m, \alpha_m) = 2$ for $m = N{+}1, 2, \cdots, N{+}M$ (see also (4.16)). In these rules, the length-squared of the long root should be regarded as $(\alpha_{\rm long}, \alpha_{\rm long}) = -4$. Moreover, the BPS condition also becomes more complicated than the one in the ordinary setup. This can be easily understood from the heterotic dual viewpoint. Consider the heterotic $OSp(32+2n|2n)$ string on T^2. The BPS condition requires that the right-moving sector is in the ground state. Thus from (4.14) we immediately find

$$(\alpha_{\rm long}, \alpha_{\rm long}) \leq 2 - 2\Delta_{\rm oscillator}, \tag{6.2}$$

where we assumed that there are no momenta in the compactified directions. When the equality is saturated, the mode becomes massless. For the long root $(\alpha_{\rm long}, \alpha_{\rm long}) = -4$, the left-moving part of the vertex operator looks like

$$J^{\alpha_{\rm long}} = e^{2i\tilde{\varphi}_N} \partial\xi \partial^2 \xi. \tag{6.3}$$

Thus this includes the oscillator excitation of $\Delta_{\rm oscillator} = 3$ and it indeed becomes massless.

It is possible to obtain a superalgebra counterpart of E_n by adding ghost D7-branes in the E_n 7-brane configuration. Such superalgebras are, as we have seen from the heterotic string viewpoint, infinite dimensional because the Cartan matrix A_{ij} is no longer positive definite. Such algebras are called indefinite superalgebras. One simple example is obtained from the D_5 Dynkin diagram (=7-brane configuration) by adding a fermionic node (=a

[28]However, it is known that the $Sp(N)$ gauge group can be realized in the presence of discrete fluxes on K3 surfaces [42], which is dual to the CHL heterotic string theory [43].

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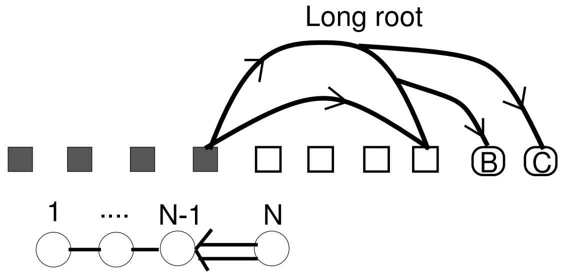

Figure 4: The description of the long root $\alpha^2_{\text{long}} = 4$ in $Sp(2N)$ in terms of a string junction. Even though a long root is not a BPS junction in ordinary string theory, it becomes BPS in our setup which includes ghost 7-branes. This is not a simple root in the superalgebra $OSp(2M|2N)$ and thus did not show up in figure 3.

ghost D7-brane) so that the Dynkin diagram becomes similar to E_6 (see e.g. [44]). It will also be interesting to see if we can obtain the affine Lie superalgebras from 7-brane configurations.

Finally, we discuss the F-theory interpretation. When we separate ghost 7-branes from D7-branes, the value of Imτ ($\tau \equiv ie^{-\phi} + \chi$) is negative near a ghost brane. This is not possible if we identify τ with the period of the torus fiber in the elliptically fibered K3 surface. This suggests that we need to consider F-theory on a sort of generalized K3 surface. We expect that such a manifold would be a supermanifold with complex superdimension two.[29] We encounter a similar situation when we examine the type IIA/heterotic duality as we discuss in the final section.

7. Discussions

7.1 Isolated ghost D-branes

When M ghost D-branes are on top of $N \geq M$ ordinary D-branes with trivial gauge backgrounds, the effects of ghost D-branes are completely canceled. It is clear that there is no physical pathology in this system. The physical interpretation is subtler when some background fields are turned on and ghost D-branes are not canceled by ordinary branes. For example, when ghost D-branes are separated from ordinary D-branes, the system has negative tension objects on which ghost fields appear. The Born-Infeld analysis seems to indicate that the minimum energy configuration is such that the ghost D-branes are moving at the speed of light and the D-branes are static. This configuration is non-supersymmetric and may decay into some other background.

On the other hand, we have seen several interesting properties that may turn ghost D-branes into a useful notion in string theory. First of all, they preserve the same supercharges as ordinary D-branes, so the combined system is BPS. Second, in the compactified heterotic string dual discussed in section 4, we found the degrees of freedom corresponding to turning on the Wilson line. They are T-dual to moving ghost D8-branes. We believe that

[29]Some of earlier works on supermanifolds are, e.g., [45, 46, 7]. More recently supermanifolds have been discussed in the topological string theory on twistor spaces in [47 – 49].

these issues deserve further study and that they may provide us with physically important consequences.

One intriguing model would be a system of static D-branes and a ghost D-brane moving toward the D-branes. Since the transverse scalars of the ghost D-brane has the wrong signs in their kinetic terms, this system would describe a ghost condensation. Also if we assume that the D-branes are far apart from the ghost D-brane, the influence of the ghost D-brane on an observer sitting on the D-branes will be tiny, as the bulk theory is described by the ordinary superstring theory. Keeping in mind the non-unitarity and instabilities of the ghost fields, it may still be interesting to explore applications of ghost branes to various phenomenological purposes. For example, the possible role of ghost fields (a.k.a. phantom matter) as dark energy has been discussed in cosmology [50]. Ghosts also appear in [51]. An even closer example may be [52], where the Z_2 symmetry which flips the sign of energy is proposed in the matter sector as a possible resolution of the cosmological constant problem.

7.2 IIA/Heterotic Duality

It is well-known that type II string theory on K3 is dual to heterotic string theory on T^4 [53] [54] [55] . In this correspondence, the moduli space of $N = (4, 4)$ $\hat{c} = 2$ conformal field theory on the K3 surface

$$SO(4, 20, \mathbb{Z})\Big\backslash SO(4, 20)\Big/ SO(4) \times SO(20). \tag{7.1}$$

is equivalent to the one for the heterotic string on T^4. Under this duality, a fundamental heterotic string is mapped to an NS5-brane wrapped on the entire K3 surface in type IIA string theory.

We expect that a similar duality will also hold for our heterotic string. This leads to the conjecture that the $OSp(32 + 2n|2n)$ heterotic string on T^4 is equivalent to type IIA string theory on a certain manifold whose sigma model leads to a $\hat{c} = 2$ SCFT with the following moduli space

$$SO(4 + n, 20 + n, \mathbb{Z})\Big\backslash SO(4 + n, 20 + n)\Big/ SO(4) \times SO(n, 20 + n), \tag{7.2}$$

as is clear from (4.26).

Since K3 is known to be the unique compact and simply connected Ricci flat manifold with complex dimension two, we probably need to consider a supermanifold with super dimension two (i.e., bosonic dim. − fermionic dim.=2) in the same sprit as in the heterotic string side. Notice that the same manifold can occur in the F-theory description discussed in section 5. Moreover, a supermanifold version of ALE spaces can naturally arise by considering the T-dual of the system with NS5-branes and ghost NS5-branes.

7.3 Other superalgebras

It would be interesting to ask if finite dimensional Lie superalgebras other than $U(N|M)$ and $OSp(N|M)$ can appear in some open string theory as Chan-Paton matrices. To have such an interpretation, they need to be realized as subalgebras of the supermatrices $gl(N|M)$. From the Kac classification [56] of Lie superalgebras, we find two families of such superalgebras $Q(N)$ and $P(N)$. These are called strange superalgebras and are defined by the supermatrices

$$\Phi_{Q(N)} = \begin{pmatrix} \phi & \psi \\ \psi & \phi \end{pmatrix}, \qquad \Phi_{P(N)} = \begin{pmatrix} \phi & \psi_S \\ \psi_A & -\phi^t \end{pmatrix}, \tag{7.3}$$

where ϕ and ψ are $N \times N$ bosonic and fermionic matrices,[30] respectively; ψ_S and ψ_A are the fermionic symmetric and antisymmetric matrices.

Formally, we can find a $\mathbb{Z}_2$ action h which projects a system of N D-branes and N ghost D-branes into the system that corresponds to (7.3). They are given[31] by $h_Q = -(-1)^{F_S}$ and $h_P = -\Omega(-1)^{F_s}$ (F_S is the spacetime fermion number) for $Q(N)$ and $P(N)$ respectively. This is because the action $-(-1)^{F_S}$ flips the signs of the NSNS and RR parts of a boundary state and a D-brane is mapped to a ghost D-brane (i.e. it acts as $\Phi \to \sigma_2 \Phi \sigma_2$). However, the closed string theories arising from projections by h_Q and h_P do not seem to make sense; in particular the OPEs may not close.

Other interesting superalgebras are the exceptional ones. There are two of them: $F(4)$ and $G(3)$. Since the non-simply laced Lie group F_4 can be found as part of a gauge group in the CHL string theory [43], these supergroups might somehow show up in heterotic string theory.

7.4 Future directions

There are various 'ghost' branes whose existence in string theories are predicted by dualities. These include the negative tension versions of NS5-branes, fundamental strings, M2-branes, and M5-branes. It would be interesting to study the properties of these objects.

The reduction of a system with supergroup symmetries to one with usual bosonic symmetries as described in subsection 3.1 holds not just in string theory but in general quantum field theories. It may be possible to find useful applications of this.

It is known that the perturbation theory does not appear to reduce to $SU(N-M)$ around a vacuum of the $SU(N|M)$ Chern-Simons theory [14]. More generally, one can consider a topological brane-anti brane system where some background fields are turned on and anti branes are not completely canceled. We expect that the topological string amplitudes for such a system computes some terms in the low-energy action of corresponding system of D-branes and ghost D-branes, by replacing traces with supertraces in the usual formulas. This may have some applications.

Note added. A few weeks after our paper appeared on the web, we received an interesting paper [57], where ghost D3-branes are applied to discuss a version of AdS/CFT correspondence. There, the holographically dual theory is the 4D $N = 4$ $SU(N|N)$ super Yang-Mills theory. As pointed out in that paper, non-superymmetric gauge theories with

[30] Strictly speaking the mathematical definition of $Q(N)$ and $P(N)$ requires that ϕ and ψ are traceless to make the algebra simple. We ignore this condition here as we usually do when discussing the $U(N)$ gauge symmetry on N Dp-branes.

[31] Here we have type II strings in mind. In the bosonic string case, we can just set $F_S = 0$.

the supergroup $SU(N|N)$ had been used to realize gauge invariant regularization of $SU(N)$ theories in a series of work starting with [58].

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A. Definition of Lie superalgebras

Lie superalgebras [56] are defined by replacing the commutation relations in the definition of usual Lie algebras with the $\mathbb{Z}_2$ graded ones such that (anti-)commutators satisfy

$$[X,Y] = -(-1)^{|X||Y|}[Y,X]; \qquad [X,[Y,Z]] = [[X,Y],Z] + (-1)^{|X||Y|}[Y,[X,Z]], \tag{A.1}$$

where $|X|$ denotes the fermion number of X, i.e. $|X| = 0$ if X is even (or bosonic) and $|X| = 1$ if X is odd (or fermionic). Finite dimensional Lie superalgebras were classified by Kac [56]. Some of them can be expressed by supermatrices. For more extensive reviews, refer to [15 – 17].

A.1 Preliminaries

We express a supermatrix X in terms of bosonic A, B or fermionic C, D submatrices

$$X = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in gl(N|M). \tag{A.2}$$

We define supertrace Str and superdeterminant Sdet of X by

$$\mathrm{Str}X = \mathrm{Tr}\,A - \mathrm{Tr}\,D, \qquad \mathrm{Sdet}X = \det(A - BD^{-1}C)\cdot\det(D)^{-1}. \tag{A.3}$$

They satisfy

$$\mathrm{Str}(MN) = \mathrm{Str}(NM), \quad \mathrm{Sdet}(MN) = \mathrm{Sdet}(M)\mathrm{Sdet}(N), \quad \mathrm{Sdet}(e^M) = e^{\mathrm{Str}M}. \tag{A.4}$$

In order to be consistent with these, we work with a supertransposed matrix

$$X^{\tilde{T}} = \begin{pmatrix} A^T & C^T \\ -B^T & D^T \end{pmatrix}$$

instead of a usual transposed matrix X^T. We can indeed show that

$$\mathrm{Str}(X^{\tilde{T}}) = \mathrm{Str}X, \quad \mathrm{Sdet}(X^{\tilde{T}}) = \mathrm{Sdet}X, \tag{A.5}$$

and

$$(XY)^{\tilde{T}} = Y^{\tilde{T}}X^{\tilde{T}}. \tag{A.6}$$

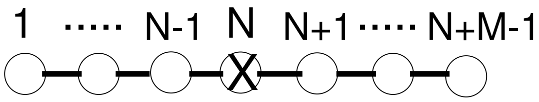

Figure 5: The Dynkin diagram of the superalgebra $SU(N|M)$ (also called $A(N-1|M-1)$). Each node, labeled by an integer m, represents a simple root α_m $(m = 1, 2, \cdots, N+M)$. The gray node $\otimes$ is a fermionic root α_N and its length is zero. The other roots represented by white nodes $\bigcirc$ have length-squared $(\alpha_m, \alpha_m) = \pm 2$.

Note that $\tilde{T}$ is not a $\mathbb{Z}_2$ action, but a $\mathbb{Z}_4$ action. In fact we can see

$$(X^{\tilde{T}})^{\tilde{T}} = KXK = \begin{pmatrix} A & -B \\ -C & D \end{pmatrix}, \tag{A.7}$$

where K denotes

$$K = \begin{pmatrix} I_N & 0 \\ 0 & -I_M \end{pmatrix}. \tag{A.8}$$

On the other hand, the adjoint operation remains the ordinary one

$$X^\dagger = (X^T)^*, \tag{A.9}$$

which involves T rather than $\tilde{T}$. This satisfies

$$(XY)^\dagger = Y^\dagger X^\dagger, \quad \text{and} \quad (X^\dagger)^\dagger = X. \tag{A.10}$$

We call a supermatrix X hermitian if $X^\dagger = X$.

A.2 $SU(N|M)$

An element of the Lie superalgebra $SU(N|M)$ (also called $A(N-1|M-1)$) is a hermitian supermatrix Φ whose supertrace is zero: $\mathrm{Str}\Phi = 0$. It is divided into bosonic and fermionic elements

$$\Phi = \begin{pmatrix} \phi^1_{ij} & \psi_{ia} \\ \psi^*_{ia} & \phi^2_{ab} \end{pmatrix}, \tag{A.11}$$

where $1 \leq i, j \leq N$ and $N+1 \leq a, b \leq N+M$. The matrices ϕ^1 and ϕ^2 are bosonic and belong to the Lie algebras $U(N)$ and $U(M)$, with the constraint $\sum_i \phi^1_{ii} - \sum_a \phi^2_{aa} = 0$. The complex matrix ψ is fermionic. Fig.5 is the Dynkin diagram of $SU(N|M)$.

In the special case $N = M$ the algebra is not simple because the element I_{2N} commutes with everything else. We have to take a quotient by $U(1)$ to make it simple. This is called $PSU(N|N)$.

A.3 $OSp(N|M)$

We move on to the orthosymplectic algebra $OSp(N|M)$, where M is always even. In Kac's classification, this is called $D(N/2, M/2)$ if N is even, $C(\frac{M}{2}+1)$ if $N = 2$, and $B(\frac{N-1}{2}, \frac{M}{2})$ if N is odd.

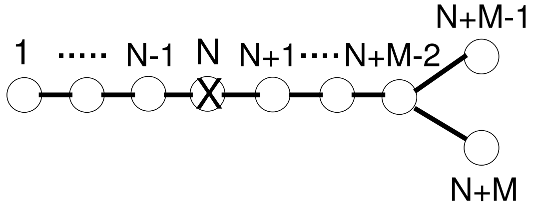

Figure 6: The Dynkin diagram of the superalgebra $OSp(2M|2N)$ (or called $D(N, M)$). Each node, labeled by an integer m, represents a simple root α_m ($m = 1, 2, \cdots, N + M$). The gray node $\otimes$ is a fermionic root α_N, whose length is zero. The other simple roots expressed by the while nodes $\bigcirc$ have length-squared $(\alpha_m, \alpha_m) = \pm 2$. The subdiagram with the simple roots $\alpha_{N+1}, \cdots, \alpha_{N+M}$ is identical to the D_M Dynkin diagram.

The $OSp(N|M)$ is defined by imposing the constraint on $SU(N|M)$, i.e.

$$\gamma \cdot \Phi + \Phi^{\tilde{T}} \cdot \gamma = 0, \tag{A.12}$$

where γ is defined by

$$\gamma = \begin{pmatrix} I_N & 0 \\ 0 & \eta_M \end{pmatrix}, \tag{A.13}$$

where η_M is the $M \times M$ matrix

$$\eta_M = iJ_M = \begin{pmatrix} 0 & iI_{M/2} \\ -iI_{M/2} & 0 \end{pmatrix}. \tag{A.14}$$

Solutions to (A.12) are the superalgebra elements of $OSp(N|M)$ and can be written as

$$\Phi = \begin{pmatrix} \phi^1_{ij} & -(\psi^T \eta)_{ia} \\ \psi_{ai} & \phi^2_{ab} \end{pmatrix}. \tag{A.15}$$

The first bosonic part ϕ^1 satisfies the $O(N)$ projection

$$(\phi^1)^T = -\phi^1, \tag{A.16}$$

while the second one ϕ^2 satisfies the $Sp(M)$ (or $USp(M/2)$) projection

$$(\phi^2)^T = -\eta \cdot \phi^2 \cdot \eta. \tag{A.17}$$

Due to the hermiticity condition, the fermionic part obeys

$$\psi^* = \eta\psi. \tag{A.18}$$

Fig.6 is the Dynkin Diagram of the superalgebra $OSp(2M|2N) = D(N, M)$.

A.4 Other Lie superalgebras

There are many other superalgebras in Kac's classification. Here we summarize them. In general, Lie superalgebras fall into two classes: classical Lie superalgebras and Cartan type superalgebras.

In addition to $SU(N|M)$ and $OSp(N|M)$ (also called $A(n,m)$, $B(n,m)$, $C(n+1)$, $D(n,m)$ in Kac's classification as we mentioned), the classical Lie superalgebras include the exceptional ones called $F(4)$ and $G(3)$. They have as the bosonic parts the Lie algebras of $SO(7)\times SU(2)$ and $G_2\times SU(2)$, respectively. Also it is known that $D(2,1)$ has a continuous parameter α and is called $D(2,1;\alpha)$. Furthermore, the classical superalgebras also include the so-called strange superalgebras denoted by $Q(n)$ and $P(n)$.

Finally, there are four families of Cartan type superalgebras called $W(n), S(n), \tilde{S}(n)$ and $H(n)$. They are defined as (sub)algebras of the vector fields on the n dimensional flat fermionic manifold, whose coordinates are given by n Grassmann numbers $(\theta_1, \theta_2, \cdots, \theta_n)$.

B. Proofs of cancellation in $U(N|M)$ and $OSp(N|M)$ super matrix models

B.1 Proofs by Virasoro constraints

The $U(N|M)$ supermatrix model is defined by the action

$$S = -\mathrm{Str}V(\Phi), \quad V(\Phi) = \sum_{n\geq 1} c_n \Phi^n \tag{B.1}$$

and the matrix integral

$$Z[U(N|M)] = \int d\Phi\ e^{-S(\Phi)}. \tag{B.2}$$

We can derive an infinite number of partial differential equations which should be satisfied by the partition function $Z[U(N|M)]$ in the form of Virasoro constraints. They can be found by shifting the supermatrix as

$$\Phi \to \Phi' = \Phi + \epsilon \left(\frac{1}{z-\Phi}\right). \tag{B.3}$$

To compute the Jacobian, let us consider the variation

$$\delta\Phi' = \delta\Phi + \epsilon \textstyle\sum_{k=0}^{\infty} \sum_{l=0}^{k} \frac{1}{z^{k+1}} \Phi^l \delta\Phi \Phi^{k-l}. \tag{B.4}$$

Then the superJacobian reads

$$J = 1 + \epsilon \left(\mathrm{Str}\frac{1}{z-\Phi}\right)^2. \tag{B.5}$$

By combining the superJacobian and the variation of the action, we obtain the loop equation

$$\left\langle \left(\mathrm{Str}\frac{1}{z-\Phi}\right)^2 + \mathrm{Str}\frac{V'(\Phi)}{z-\Phi} \right\rangle = 0.$$

The $\mathcal{O}(1/z^{k+2})$ term reads

$$\left\langle \sum_{l=0}^{k} \mathrm{Str}\Phi^l \mathrm{Str}\Phi^{k-l} + \sum_n n c_n \mathrm{Str}\Phi^{n+k} \right\rangle = 0,$$

which can also be written as

$$0 = L_k Z \equiv \left(2(N-M)\partial_k + \sum_{l=1}^{k-1} \partial_l \partial_{k-l} + \sum_{n=0}^{\infty} n c_n \partial_{k+n} \right) Z.$$

for $k \geq 1$ and

$$\begin{aligned} 0 &= L_0 Z \equiv \left((N-M)^2 + \sum_{n=0}^{\infty} n c_n \partial_n \right) Z, \\ 0 &= L_{-1} Z \equiv \left(c_1 (N-M) + \sum_{n=0}^{\infty} (n+1) c_{n+1} \partial_n \right) Z. \end{aligned} \tag{B.6}$$

These differential operators satisfy the Virasoro algebra

$$[L_k, L_l] = (k-l) L_{k+l}.$$

The fact the Virasoro generators depend on N and M only through the combination $N-M$ suggests that the dynamics of the supermatrix model is identical to that of the $U(N-M)$ matrix model.

We move on to the OSp case. Consider the supermatrix model

$$Z[OSp(N|M)] = \int d\Phi e^{\mathrm{Str} V(\Phi)}, \quad V(\Phi) = \sum_n c_n \Phi^{2n}. \tag{B.7}$$

The integral is over the Lie superalgebra of $OSp(N|M)$. We include only even powers of Φ in V because the supertrace vanishes on odd powers. Consider the following infinitesimal variation in the Lie superalgebra direction:

$$\Phi \to \Phi' = \Phi + \epsilon \left(\frac{1}{z - \Phi} \right)_{\mathrm{odd}} = \Phi + \epsilon \frac{\Phi}{z^2 - \Phi^2}.$$

One can show[32] that the superJacobian to order ϵ is

$$J = 1 + \frac{\epsilon z^2}{2} \left(\mathrm{Str} \frac{1}{z^2 - \Phi^2} \right)^2 - \frac{\epsilon}{2} \mathrm{Str} \frac{1}{z^2 - \Phi^2}$$

and we obtain the Ward identity

$$\left\langle \left(\mathrm{Str} \frac{z}{z^2 - \Phi^2} \right)^2 - \mathrm{Str} \frac{1}{z^2 - \Phi^2} + 2 \mathrm{Str} \frac{\Phi V'(\Phi)}{z^2 - \Phi^2} \right\rangle = 0.$$

This is equivalent to the Virasoro constraints

$$0 = L_k Z \equiv \left(\frac{1}{2}(N - M - \frac{1}{2}) \partial_k + \frac{1}{4} \sum_{l=1}^{k-1} \partial_l \partial_{k-l} + \sum_{n=0}^{\infty} n c_n \partial_{k+n} \right) Z.$$

for $k \geq 1$ and

$$0 = L_0 Z \equiv \left(\frac{1}{4}(N-M)(N-M-1) + \sum_{n=0}^{\infty} n c_n \partial_n \right) Z.$$

The appearance N and M through $N-M$ indicates that the $OSp(N|M)$ model reduces to the $SO(N-M)$ model.

[32] $\xi^{\mu\nu} := (\Phi\gamma^{-1})^{\mu\nu}$ are 'anti-symmetric': $\xi^{\nu\mu} = -(-1)^{|\mu||\nu|}\xi^{\mu\nu}$. This property makes it easy to consider independent components.

B.2 Perturbative proof of cancellation in the $U(N|M)$ matrix model

We consider correlation functions in the $U(N|M)$ supermatrix model (B.1). The propagators can be written as (we set $c_2 = 1$ by rescaling)

$$\begin{aligned}
\langle \phi^1_{ij}\phi^1_{kl}\rangle &= \delta_{il}\delta_{jk},\\
\langle \phi^2_{ij}\phi^2_{kl}\rangle &= -\delta_{il}\delta_{jk},\\
\langle \psi_{ij}\psi^*_{kl}\rangle &= -\delta_{il}\delta_{jk}.
\end{aligned} \tag{B.8}$$

We can also write them in a compact way

$$\langle \Phi_{\mu\nu}\Phi_{\rho\sigma}\rangle = \delta_{\mu\sigma}\delta_{\nu\rho}(-1)^{|\nu|}. \tag{B.9}$$

In this notation the supertrace is given by $\mathrm{Str}\Phi = \sum_\mu (-1)^{|\mu|}\Phi_{\mu\mu}$.

Below we follow the arguments in [29] to show that any correlation function in this matrix model only depends on $N - M$. Since we can perform perturbative expansions of interaction terms, we have only to examine correlation functions in the free theory (i.e. $c_n = 0$ for $n \geq 3$). To make the fermionic nature of the indexes manifest, we rewrite $\Phi_{\mu\nu}$ as $\alpha_\mu\bar{\alpha}_\nu$. Then α_i are bosonic while α_a are fermionic. Consider the operator product $\mathrm{Str}\Phi^3\mathrm{Str}\Phi^3$ which can be expressed as

$$\begin{aligned}
&(-1)^{|\mu|}\alpha_\mu\bar{\alpha}_\nu\alpha_\nu\bar{\alpha}_\rho\alpha_\rho\bar{\alpha}_\mu \cdot (-1)^{|\xi|}\alpha_\xi\bar{\alpha}_\eta\alpha_\eta\bar{\alpha}_\sigma\alpha_\sigma\bar{\alpha}_\xi\\
&\quad = (\bar{\alpha}_\mu\alpha_\mu)(\bar{\alpha}_\nu\alpha_\nu)(\bar{\alpha}_\rho\alpha_\rho)\cdot(\bar{\alpha}_\xi\alpha_\xi)(\bar{\alpha}_\eta\alpha_\eta)(\bar{\alpha}_\sigma\alpha_\sigma).
\end{aligned} \tag{B.10}$$

Then we take the Wick contractions using the propagator (B.9). We concentrate on a particular contraction. Then by moving only pairs of $(\bar{\alpha}\alpha)$ we can always divide (B.10) into several parts such that in each part the contraction is taken successively following the array of α_μs

$$\begin{aligned}
&\left[(\bar{\alpha}_\mu\widehat{\alpha_\mu)(\bar{\alpha}_{\mu'}}\widehat{\alpha_{\mu'})\cdot}\cdots\right]\cdot\left[(\bar{\alpha}_\nu\widehat{\alpha_\nu)(\bar{\alpha}_{\nu'}}\widehat{\alpha_{\nu'})\cdot}\cdots\right]\cdots\\
&\quad = (-1)^{\#\text{fermionic loops}}\cdot\left[\widehat{\alpha_\mu\bar{\alpha}_{\mu'}}\widehat{\alpha_{\mu'}\cdot}\cdots\bar{\alpha}_\mu\right]\cdot\left[\widehat{\alpha_{\nu'}\bar{\alpha}_\nu}\widehat{\alpha_{\nu}\cdot}\cdots\bar{\alpha}_{\nu'}\right]\cdots,
\end{aligned} \tag{B.11}$$

where #fermionic loops denotes the number of loops where the sum with respect to $a = N+1, \cdots, N+M$ is taken. The contraction is denoted by $\widehat{\alpha\bar{\alpha}}$. Now evaluate the contraction using the propagator (B.9). We do this as follows

$$\widehat{\alpha_\mu\bar{\alpha}_\lambda}\widehat{\alpha_\rho\bar{\alpha}_\nu} = (-1)^{|\nu|}\langle\Phi_{\mu\nu}\Phi_{\rho\lambda}\rangle = \delta_{\mu\lambda}\delta_{\nu\rho}. \tag{B.12}$$

Indeed it is easy to see that we can simply replace each contraction $\widehat{\alpha_\mu\bar{\alpha}_\nu}$ with the $\delta_{\mu\nu}$. In this way the sum over each Feynman diagram looks like

$$\begin{aligned}
&\langle \mathrm{Str}\Phi^{n_1}\mathrm{Str}\Phi^{n_2}\cdots\rangle\\
&= \sum_{\text{All diagrams}}\left[\sum_{\mu_1}(-1)^{|\mu_1|}\delta_{\mu_1\mu_1}\right]\cdot\left[\sum_{\mu_2}(-1)^{|\mu_2|}\delta_{\mu_2\mu_2}\right]\cdots\left[\sum_{\mu_L}(-1)^{|\mu_L|}\delta_{\mu_L\mu_L}\right]\\
&= \sum_{\text{All diagrams}}(N-M)^L,
\end{aligned} \tag{B.13}$$

where L is the number of loops in each Feynman diagram. Thus we have shown (3.2). Note that the dependence on N and M only through $N-M$ holds for individual Wick contractions. This shows that the $U(N|M)$ matrix model reduces to the $U(N-M)$ matrix model with the same action.

So far we have been considering a system with $U(N|M)$ gauge symmetry, or a system where the fields transform in the adjoint representation of $U(N|M)$. We can also consider a situation with a $U(N'|M')$ flavor symmetry. This can be modeled by bi-fundamental fields $Q_{\mu m}, \bar{Q}_{m\mu}$, where μ and m are vector indices for $U(N|M)$ and $U(N'|M')$, respectively. The propagator is

$$\langle Q_{\mu m}\bar{Q}_{n\nu}\rangle = (-1)^{|m|}\delta_{\mu\nu}\delta_{mn}.$$

By writing

$$Q_{\mu m} = \beta_\mu\bar{\gamma}_m, \quad \bar{Q}_{m\mu} = \gamma_m\bar{\beta}_\mu,$$

the argument above goes through and shows that the results of perturbative computations depend on N' and M' only through $N'-M'$. Thus a system with $N'|M'$ quarks is equivalent to a system with $N'-M'$ quarks.

B.3 Perturbative proof of cancellation in the $OSp(N|M)$ matrix Model

Next we consider the perturbative expansion of (B.7). The supermatrix takes the form (A.15). The propagators are given by (we set $c_2 = 1/4$)

$$\begin{aligned}
\langle\phi^1_{ij}\phi^1_{kl}\rangle &= \delta_{il}\delta_{jk} - \delta_{ik}\delta_{jl}, \\
\langle\phi^2_{ab}\phi^2_{cd}\rangle &= -\delta_{ad}\delta_{bc} - \eta_{ac}\eta_{bd}, \\
\langle\psi_{ai}\psi_{bj}\rangle &= -\delta_{ij}\eta_{ab}, \\
\langle(-\psi^T\eta)_{ia}\psi_{bj}\rangle &= -\delta_{ab}\delta_{ij}.
\end{aligned} \tag{B.14}$$

We can summarize (B.14) as

$$\langle\Phi_{\mu\nu}\Phi_{\rho\sigma}\rangle = (-1)^{|\nu|}\delta_{\mu\sigma}\delta_{\nu\rho} - (-1)^{|\mu||\nu|}\gamma_{\sigma\nu}\gamma_{\mu\rho}, \tag{B.15}$$

where $\gamma_{\mu\nu}$ is defined in (A.13). It is easy to see that (B.15) is consistent with the projection (A.12).

To show (3.3), let us first compare this with the previous $U(N|M)$ case. The only difference is the second term in the propagator (B.15). We can apply the argument in the $U(N|M)$ case to contractions involving only the first term. The total expression of the correlation function can be obtained by replacing some of the contractions $(-1)^{|\nu|}\delta_{\mu\sigma}\delta_{\nu\rho}$s with the second one $-(-1)^{|\mu||\nu|}\gamma_{\sigma\nu}\gamma_{\mu\rho}$s.

Remember that the previous result in the $SU(N|M)$ case holds for each Wick contraction, which looks like

$$[(-1)^{|\mu_1|}\delta_{\mu_1\mu_2}\delta_{\mu_2\mu_3}\cdots\delta_{\mu_A\mu_1}]\cdot[(-1)^{|\nu_1|}\delta_{\nu_1\nu_2}\delta_{\nu_2\nu_3}\cdots\delta_{\nu_B\nu_1}]\cdot[\cdot]\cdots. \tag{B.16}$$

Let us replace one of the propagators with the one in the second term. Without losing generality we can replace $\delta_{\mu_1\mu_2}\delta_{\nu_1\nu_2}$ with $-(-1)^{|\mu_2|+|\mu_1||\mu_2|}\gamma_{\mu_2\nu_1}\gamma_{\mu_1\nu_2}$. Then (B.16) is changed into

$$-(-1)^{|\mu_1|+|\nu_1|}\cdot(-1)^{|\mu_2|+|\mu_1||\mu_2|}\gamma_{\mu_2\nu_1}\gamma_{\mu_1\nu_2}\delta_{\mu_1\mu_2}\delta_{\nu_1\nu_2} = (-1)^{|\mu_1|}\delta_{\mu_1\mu_1}, \tag{B.17}$$

where we have employed the identity $\gamma_{\mu\nu} = (-1)^{|\mu||\nu|}\gamma_{\nu\mu}$ and $\gamma^2 = 1$. Therefore we have shown that this result depends only on $N-M$ again, though the power of $N-M$ is reduced by one. More general cases can be handled by induction. This completes the proof of (3.3).

B.4 Anomaly cancellation

Here we show that the results (3.7) can be found by just replacing Tr in the results for $O(N-M)$ with the supertrace Str. When we compute $\mathrm{Tr}_{ad}[F^{2m}]$ by taking only the first term in (3.6) into account, then it is obvious that we obtain $\mathrm{Str}[1]\mathrm{Str}[F^{2m}] = (N-M)\mathrm{Str}[F^{2m}]$. In order to incorporate other terms we can replace $\delta_{\lambda\nu}t^A_{\mu\rho}$ with them. For example, let us start with the desired expression (i.e. written in terms of supertrace) in a general from

$$(-1)^{|\lambda|}\delta_{\lambda\nu}M_{\nu\lambda}(-1)^{\mu}t^A_{\mu\rho}N_{\rho\mu}\cdots, \tag{B.18}$$

for some matrices M and N. If we replace $\delta_{\lambda\nu}t^A_{\mu\rho}$ with the second term, then we find

$$(-1)^{|\lambda|}M_{\nu\lambda}(-1)^{\mu}N_{\rho\mu}(-1)^{(|\mu|+|\lambda|)(|\lambda|+|\nu|)+|\lambda|+|\mu|}\delta_{\mu\rho}t^A_{\lambda\nu} = -(-1)^{|\mu|}N_{\mu\mu}\cdot(-1)^{|\lambda|}t^A_{\lambda\nu}M_{\nu\lambda}. \tag{B.19}$$

Therefore again we can find the desired form. For other two terms, we can proceed in the same way remembering the identities used in (B.17). In this way (3.7) is proved by induction.

C. Some properties of affine super algebras

A Lie superalgebra has its affine extension just like an ordinary Lie algebra. The affine super Kac-Moody algebras at level-k is defined by (4.6) with $\mathrm{Str}[t^At^B]$ replaced by $k\mathrm{Str}[t^At^B]$. It defines a conformal field theory[33] via the Sugawara construction [35] [36].

In general the central charge of the corresponding Virasoro algebra (obtained from the Sugawara construction) for the Lie supergroup G is given by

$$c = \frac{k\cdot \mathrm{sdim}G}{k+h}, \tag{C.1}$$

where k is the level and $\mathrm{sdim}G = \mathrm{dim}G_B - \mathrm{dim}G_F$ is the super dimension of the supergroup G. h is the dual Coxeter number.

In the affine $SU(N|M)$ algebra case, we find

$$\mathrm{sdim}G = (N^2+M^2-1) - 2NM = (N-M+1)(N-M-1), \tag{C.2}$$

and

$$h = N-M. \tag{C.3}$$

[33]Notice that in these supergroup cases, sometimes it is possible to have conformal field theories even if we do not turn on the WZW interaction terms. Such a model is called the principal chiral model and it is indeed conformal when $G = PSU(N|N)$ and $G = OSp(M+2|M)$ i.e. when h vanishes [14] [32]. Such a model may also be relevant to $N=2$ string theory as a holographic description.

Therefore its central charge is given by

$$c = \frac{k(N-M+1)(N-M-1)}{k+N-M}. \tag{C.4}$$

This only depends on the difference $N-M$ as expected. Notice that at $k=1$ it leads to $c = N-M-1$ as can be understood from the vertex operator construction.

In the $OSp(N|M)$ case, we obtain

$$\mathrm{sdim}G = \frac{N(N-1)}{2} + \frac{M(M+1)}{2} - NM = \frac{1}{2}(N-M)(N-M-1), \tag{C.5}$$

and

$$h = N - M - 2. \tag{C.6}$$

Therefore the central charge is given by

$$c = \frac{k(N-M)(N-M-1)}{2(k+N-M-2)}. \tag{C.7}$$

It again only depends on the difference $N-M$ as expected. At $k=1$ we find $c = \frac{N-M}{2}$. For example, the affine CFT $OSp(32+2n|2n)$, the central charge remains the same as $SO(32)$. In particular it is $c=16$ at level one $k=1$.

D. Momenta in toroidal compactification of the $OSp(32+2n|2n)$ or $E(8+\frac{n}{2},\frac{n}{2}) \times E(8+\frac{n}{2},\frac{n}{2})$ heterotic string

Here we derive the formulas (4.25) for the momenta by generalizing the arguments in [59].

We consider the compactification of the $OSp(32+2n|2n)$ or $E(8+\frac{n}{2},\frac{n}{2}) \times E(8+\frac{n}{2},\frac{n}{2})$ heterotic string on T^d and turn on metric, B-field, and Wilson lines as in subsection 4.5. The world-sheet theory contains free bosons $X^{\mu=1,\cdots,d}$ with radius R, left-moving free bosons $X^{k=n+1,\cdots,16+2n}, X^{l=1,\cdots,n}$ with radius $\sqrt{2\alpha'}$.[34] There are also n pairs of $\eta\xi$ fermions of conformal weights 1 and 0. We can ignore these fermions here. The world-sheet theory is described by the following world-sheet action:

$$\begin{aligned} S = \frac{1}{4\pi}\int d\tau \int_0^{2\pi} d\sigma \Bigg[&\frac{1}{\alpha'}(g_{\mu\nu}\partial_\alpha X^\mu \partial^\alpha X^\nu - \epsilon^{\alpha\beta}B_{\mu\nu}\partial_\alpha X^\mu \partial_\beta X^\nu \\ &+\partial_\alpha X^k \partial^\alpha X^k - \partial_\alpha X^l \partial^\alpha X^l) - A^k_\mu \epsilon^{\alpha\beta}\partial_\alpha X^k \partial_\beta X^\mu + A^l_\mu \epsilon^{\alpha\beta}\partial_\alpha X^l \partial_\beta X^\mu \Bigg] \end{aligned}$$

together with the constraints $(\partial_\tau - \partial_\sigma)X^k = 0$ and $(\partial_\tau - \partial_\sigma)X^l = 0$. Here $\epsilon^{\tau\sigma} = -\epsilon^{\sigma\tau} = 1$. We take the radius for X^μ to be R. The radius for X^k and X^l is $\sqrt{2\alpha'}$, i.e., the free fermion radius. X^l are the free bosons that bosonize the spin-$\frac{1}{2}$ β-γ systems, and appear in the kinetic terms with the wrong sign.

[34]In the notation of subsection 4.3, $X^k = \sqrt{\frac{\alpha'}{2}}\varphi^{k-n}, \quad X^l = \sqrt{\frac{\alpha'}{2}}\tilde{\varphi}^l$.

The constraints are second class. To canonically quantize the system, we need to use the Dirac bracket to take the constraints into account. The canonical momenta are

$$\begin{aligned}
P_\mu(\sigma) &= \frac{1}{2\pi\alpha'}(g_{\mu\nu}\dot{X}^\nu - B_{\mu\nu}\partial_\sigma X^\nu + A^k_\mu \partial_\sigma X^k - A^l_\mu \partial_\sigma X^l),\\
P_k(\sigma) &= \frac{1}{2\pi\alpha'}(\dot{X}^k - A^k_\mu \partial_\sigma X^\mu),\\
P_l(\sigma) &= \frac{1}{2\pi\alpha'}(-\dot{X}^l + A^l_\mu \partial_\sigma X^\mu).
\end{aligned}$$

The Dirac brackets among them turn out to be

$$\begin{aligned}
\{P_\mu(\sigma), P_{\mu'}(\sigma')\}_{\rm DB} &= \frac{1}{4\pi\alpha'}(A^k_\mu A^k_{\mu'} - A^l_\mu A^l_{\mu'})\partial_\sigma\delta(\sigma-\sigma'),\\
\{P_\mu(\sigma), P_k(\sigma')\}_{\rm DB} &= \frac{1}{4\pi\alpha'}A^k_\mu\partial_\sigma\delta(\sigma-\sigma'),\\
\{P_\mu(\sigma), P_l(\sigma')\}_{\rm DB} &= -\frac{1}{4\pi\alpha'}A^l_\mu\partial_\sigma\delta(\sigma-\sigma'),\\
\{P_k(\sigma), P_{k'}(\sigma')\}_{\rm DB} &= \frac{\delta_{kk'}}{4\pi\alpha'}\partial_\sigma\delta(\sigma-\sigma'),\\
\{P_l(\sigma), P_{l'}(\sigma')\}_{\rm DB} &= -\frac{\delta_{ll'}}{4\pi\alpha'}\partial_\sigma\delta(\sigma-\sigma'),\\
\{P_k(\sigma), P_l(\sigma')\}_{\rm DB} &= 0.
\end{aligned}$$

The Dirac brackets involving X^μ are equal to the Poisson brackets. If we take the combination

$$P'_\mu(\sigma) = P_\mu(\sigma) - A^k_\mu P_k(\sigma) - A^l_\mu P_l(\sigma),$$

P'_μ, P_k, and P_l commute among themselves and are the momenta that are truly canonically conjugate to X^μ, X^k, and X^l in the presence of the constraints. Let x^μ, x^k, and x^l be the zero-modes of X^μ, X^k, and X^l. The momenta canonically conjugate the zero-modes are quantized in units of the inverse radii. Let us write

$$\int P'_\mu d\sigma = \frac{n_\mu}{R}, \quad \int P_k d\sigma = \frac{2q_k}{\sqrt{2\alpha'}}, \quad \int P_l d\sigma = \frac{2q_l}{\sqrt{2\alpha'}}.$$

Then n_μ is an integer while q_k and q_l are half integers taking values in the appropriate lattice defining the heterotic string. Let w^μ be the winding numbers for X^μ. Then one finds

$$\begin{aligned}
X^\mu(\tau,\sigma) &= x^\mu + \alpha' g^{\mu\nu}\left[\frac{n_\nu}{R} + B_{\nu\rho}Rw^\rho - q_k A^k_\nu - q_l A^l_\nu - \frac{w^\rho}{2}(A^k_\rho A^k_\nu - A^l_\rho A^l_\nu)\right]\tau\\
&\quad + Rw^\mu\sigma + \text{oscillators},\\
X^k(\tau,\sigma) &= x^k + \sqrt{\frac{\alpha'}{2}}(q_k + A^k_\mu R w^\mu)(\tau+\sigma) + \text{oscillators},\\
X^l(\tau,\sigma) &= x^l + \sqrt{\frac{\alpha'}{2}}(-q_l + A^l_\mu R w^\mu)(\tau+\sigma) + \text{oscillators}.
\end{aligned}$$

From this, one can read off the momentum lattice (4.25).

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JHEP03(2006)062

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JHEP03(2006)062

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JHEP03(2006)062